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# Dynamic Models with Spatial Conditions in Revenue Management

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# Dynamic Models with Spatial Conditions in Revenue Management

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Graduate School of Science and Technology, Hirosaki University

September, 2017

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# Abstract

*Revenue Management (RM)* is a field that deals with decision-making: what product to sell to which customers at what price under fixed capacity, uncertain demand, large fixed cost and perishable products. Traditional applications in RM are airline, hotel and car rental industries. Recently, RM is applied to not only the traditional industries, but also hospitality's services or entertainment industries, such as golf courses, restaurants, casinos, theaters and etc. In the decision-making of RM, there are allocation problems to decide when to sell which product to which customers. Models for dealing with simultaneous requests is called a *dynamic model*.

This paper focuses on the spatial seats' layout and suggests three dynamic models with seats in a table and seats placed in rows such as a restaurant, theater or stadium. States of the seats in the models stand for how seats are reserved by customers. Therefore, we can consider congestion and booking position which has not been treated in RM, by the new dynamic models.

This study proposes a model with seats in tables and indicates the model's properties such as monotonicity. In addition, from the monotonicities we can see the fact that differences of departure rate among parties cause to multiply variations of optimal policy. A tendency of a range where variation of optimal policies enlarges is shown by numerical examples. From a spatial point of view, monotonicity of degree of congestion in a facility and the relation between the number of seats that a party has and the expected revenue are indicated. Finally, we mention the relationship between a problem of this model and a challenge which is integration of RM and *Customer Relationship Management (CRM)*.

Related to the seat problem, we consider a simple case where seats are set on a single line. In this paper, a model for this problem is called a *single line seats model*. This model decides which position should be allocated to arrival parties. This study shows that ends of a block of vacant seats should be allocated to a party with any size and any fare class. This feature of this model shows that

a single line seats model can be extended over traditional dynamic model [26].

Furthermore, we introduce a *choice-based seating position model*. In this model, a customer selects a position of the seat in a facility where seats are placed on multiple lines, and one seat bundles one fare class. This model is also based on a choice-based network RM model which has been intensively researched and studied for many years. We apply Choice-based Deterministic Linear Programming (CDLP) and a decomposition approximation method to approximately compute an optimal policy of the model. Solutions of the approximation methods can be efficiently obtained when customer's behavior is based on Multinomial Logit (MNL) choice model, although this model with large block of vacant seats is normally difficult to obtain optimal solutions for our proposed model. Numerical examples for the model have been conducted and the policies which are calculated from the approximation methods are estimated by Monte Carlo simulation. The solutions of this model indicate that higher revenue is able to be implemented by considering customer's choice behavior among seat positions even though it is optimal to accept all requests at any time if we do not take account of the customer's choice behavior.

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# List of Abbreviations

RM: revenue management

NRM: network revenue management

RRM: restaurant revenue management

SP: seats per person

SS: surplus seats

ExSS: expectation of the number of excess seats

CRM: customer relationship management

CDLP: choice-based deterministic linear programming

MNL: multinomial logit

# Chapter 1

# Introduction

There are many industries which have fixed capacity and large fixed cost. Examples of the industries are airline, hotel, rental car and etc. *Revenue Management (RM)* is a field that is to manage the decision-making process in the industries. There are many kinds of the decision-making in RM. Talluri and van Ryzin [38] said that basic categories of demand-management decisions in RM are structural decisions, price decisions and quantity decisions. (See 1.1.1 in [38] for detail.) Quantity decision is to decide whether to accept or reject a request, given a structure and product prices. This thesis focuses on this quantity decision.

There are a static model and a dynamic model for this quantity decision. The static model is under an assumption which is that a request with lower fare class arrives earlier. We can see generally static model and description about that in [38] and [2]. Koide and Ishii [25] presents the static model with overbooking and cancellation for hotel industry.

On the other hand, the dynamic model is to find optimal policies under an assumption which is that requests of customers with a variety of fare classes simultaneously arrive throughout booking horizon. The optimal policy indicates that what product to sell to which customer in each period. The dynamic model with seats of one type is named a single-leg model. Lee and Hersh [26] suggested several single-leg models and its property such as monotonicity. They assume that requests arrive according to Poisson process, and approximate it by discretizing a booking horizon. Subramanian et al. [36] extended Lee and Hersh's model by cancellation and overbooking. Also, Subramanian et al. [36] indicated properties of their model and how to discretize the booking horizon. El-Haber and El-Taha [11] extended the Subramanian et al.'s model to one with two types of seats, which means they connected the model with *Network Revenue Management (NRM)*.

NRM is able to manage multiple resources simultaneously. An example of a product in NRM is tickets to depart to a destination via some transit points. Such a kind of the tickets are seen in airlines or trains industries. In the trains industries, the transit points are stations. Multiple seats among the transit points are reserved at the same time when the ticket is purchased. We can see a summary of NRM in Chapter 3 of [38] and [5].

In our models, we considered a system with seats, such as a restaurant, theater or stadium. Generally, states of a system in dynamic models are presented as remaining time to deadline (or passed time from starting time to accept requests) and the number of remaining seats (or the number of booked seats). Then, the number of remaining seats is focused in this paper. If there is one kind of seats, then the state is commonly represented by natural number which includes zero. (In this paper, $\mathbb{Z}_+$ stands for the set of the positive integral number throughout.) Three cases where three seats are occupied in facilities with five seats are shown in Figure 1.

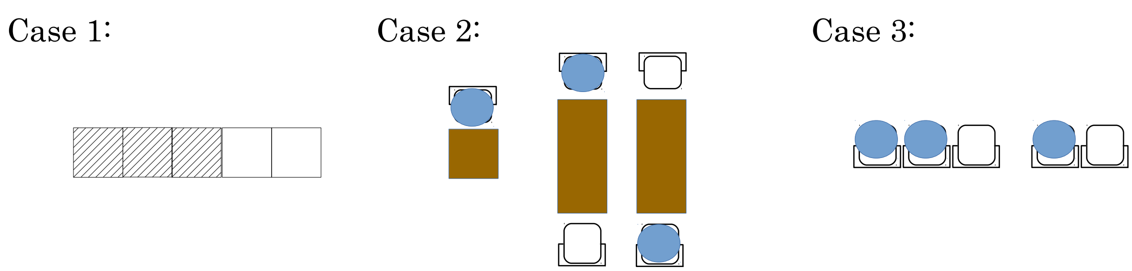


**Figure 1.1.** Three cases of a state.

Case 1, Case 2 and Case 3 in Figure 1 present the number of normal remaining seats, seats in tables and seats placed on lines, respectively. Case 2 can be regarded as a restaurant or cafeteria. Case 3 is seen in a theater or stadium. Then, it is not appropriate to treat the cases as one state. In Case 2, a manager of the facility might not recognize to able to accept an arriving couple even though two seats are not occupied. Also, in Case 3, a couple might not be willing to sit by separating in a facility such as an opera or Kabuki. For Case 3, Talluri and van Ryzin [38] actually point out this problem in p. 573. Hence, if we apply a general dynamic model to the seats with the physical feature like Case2 or Case 3, then there is a problem which is how we capture actual decision-making.

This thesis focuses on seats with this spatial condition and suggests several models which are included the physical features. Using this approach, we can find some relationships between revenue and decision-making under the physical condition such as congestion or booking position that has not ever been handled in RM.

In Chapter 2, an allocation problem for seats and tables system, which is related to Case 2, is described. The problem has already been modeled in restaurant RM. However, the models are difficult to reveal structural properties because they are too complex to analyze. This fact does not allow to link models in restaurant RM to general models in RM. Thus, we assume some conditions to simplify a model. From this way, some monotonicities of the model are obtained. From the monotonicities, we can find out that differences among departure rates of parties cause to enlarge variation of the optimal policies. Furthermore, the number of persons who are in the facility and the number of excess seats for a party are taken into the model. Finally, the relation between the model and customer relationship management is briefly mentioned.

In regard to seats placed on lines which depicted as Case 3 in Figure 1, Talluri and van Ryzin [38] state that the case is difficult from bin-packing effects if groups arrive and sit together. However, Their pointing out is intuitive because there is not a model with booking position in RM. Therefore, we cannot incorporate a customer's choice for a seating position to RM.

In Chapter 3, we suggests a simply model where a system allocates seats placed on a single line to a party. The model is named *single line seats model*. It is described that allocating ends of a vacant seats block to a party is an optimal policy if the party should be accept. This property leads to a relationship between the single line seats model and a traditional model [26]. Finally, application of the single line seats model is briefly described.

Chapter 4 then introduce a new model where a customer chooses a seating position by his/her preference. The model is called *choice-based seating position model* in this thesis. The choice-based seating position model is formulated by using an approach of single line seats model which have been already mentioned in Chapter 3. The choice-based seating position model is used to decide an offer set which is an available set of seating positions for an arriving customer. The problem is firstly modeled by dynamic programming. To find approximate solutions for the problem, we apply Choice-based Deterministic Linear Programming (CDLP) method and decomposition approximation method to the choice-based seating position model. If choice behavior of all arriving customers depends on Multinomial Logit (MNL) choice model, then we can efficiently search decision space for the optimal offer set. Finally, offer sets computed by each approximation method are estimated by numerical example.

# List of Symbols for Chapter 2

$P$: set of arriving parties

$I$: set of tables

$\overline{P}$: the number of different party sizes

$\overline{I}$: the number of different table sizes

$g_p$: party size for $p \in P$

$t_i$: table size for $i \in I$

$m_i$: the number of the table for $i \in I$

$P_i$: party set which is able to be allocated the table $i \in I$

$I_p$: table set which is able to be allocated to the party $p \in P$

$N$: size of time period

$T_0$: set of time periods $\{0, 1, \cdots, N\}$

$T_1$: set of time periods $\{1, \cdots, N\}$

$\overline{X}_i$: state space for the table $i \in I$

$X_n$: state space for a facility

$r_p^n$: expected revenue for the party $p \in P$ in $n \in T_0$

$\lambda_p^n(X)$: arrival rate for the party $p \in P$ and the state $X \in X_n$ in the period $n \in T_0$, and $\lambda_p^n(X) > 0$

$q^n_{ip}(X)$: departure rate for the party $p \in P_i$, $i \in I$, and the state $X \in X_n$ in the period $n \in T_0$

$\lambda^n_0$: probability of a null event in period $n \in T_0$

$\chi$: maximum of $|X_n|$ for $n \in T_0$

$U_n(X)$: maximum expected revenue from operating over periods $n$ to 0 in a state $X$

$\Delta^i_p U_n(X)$: opportunity cost of accepting the party $p \in P$ for the table $i \in I_p$ in $n$; $\Delta^i_p U_n(X) = U_n(X) - U_n(X + \boldsymbol{e}^i_p)$

$d$: policy vector; $d = (d_p)$ where $d_p$ is a policy for $p \in P$

$x^i$: the number of parties who sit in the table $i \in I$ for submatrix $\boldsymbol{x}_i$ of $X \in X_n, n \in T_0$

$f(X)$: the number of people in the state $X \in X_n, n \in T_0$

$SP_p(X)$: average of the number of seats of the party $p \in P$ for the state $X \in X_n, n \in T_0$

$SS_p(X)$: the number of surplus seats of the party $p \in P$ for the state $X \in X_n, n \in T_0$

$\alpha^n_p$: coefficient which converts comfort which the party $p \in P$ obtains by an extra seat in $n \in T_1$ into a price

$\hat{U}_n(X)$: maximum expected revenue which includes $\alpha^n_p, p \in P, n \in T_1$

$P_n(X)$: probability of being $X \in X_n$ in $n \in T_0 \backslash N$

$\Omega_{min}, \Omega_{max}$, and $\Omega_{SS}$: optimal policies for all time periods calculated by min-policy, max-policy, and SS-policy, respectively

$ExSS(\Omega)$: expectation of the number of excess seats which is obtained by a optimal policies $\Omega$ from $n = N$ to $n = 0$

# Chapter 2

# Seats in Tables

## 2.1 Introduction

Traditional applications of the RM are airline, hotel and car rental industries. Researches, problems, traditional models, and a glossary of revenue management for airline can be found in McGill and Ryzin [28]. Recently, for non-traditional industries, Chiang, Chen and Xu [10] reviewed recent application and techniques of RM. One of the non-traditional industries which is applicable to the theory of RM is restaurant industry. The revenue management for restaurant is called *restaurant revenue management* (RRM). In this chapter, we consider this RRM. Two problems are known in RRM. One of the problems is a table mix problem [21] which is used to decide an optimal structure for tables in a restaurant. The another problem is a party mix problem [14] which is used to obtain an optimal *seating policy* for arriving parties. There are not many researches which deal with the seating policy. Bertsimas and Shioda [6] presented some models: an integer programming, a stochastic programming, and an approximation dynamic programming model. Guerriero et al. [14] suggested a dynamic programming model with no waiting line, reservation, and meal duration by using the techniques of NRM.

These studies have focused on making models and algorithms for evaluating the expected total revenue approximately. The reason is that it is difficult to exactly calculate the seating policy. The difficulty is due largely to *the curse of dimensionality*.

Each restaurant has several kinds of tables, that is, the restaurant has several capacities. It is known that a model in the NRM is more complex than a model with single-resources. A part of reasons for the complexity is that state space enormously expands. Furthermore, in RRM, the state space enlarges more than ordinary models (seeing as an example in Sec.3.2 of [38]) in NRM. Because models in RRM must include a departure process of parties, which implies cancellation process in the airline or hotel industry. Figure 2.1 shows states for cases with no-cancellation and with cancellation. The case without the cancellation process is Case 1 and the another case with the cancellation process is Case 2 in Figure 2.1.

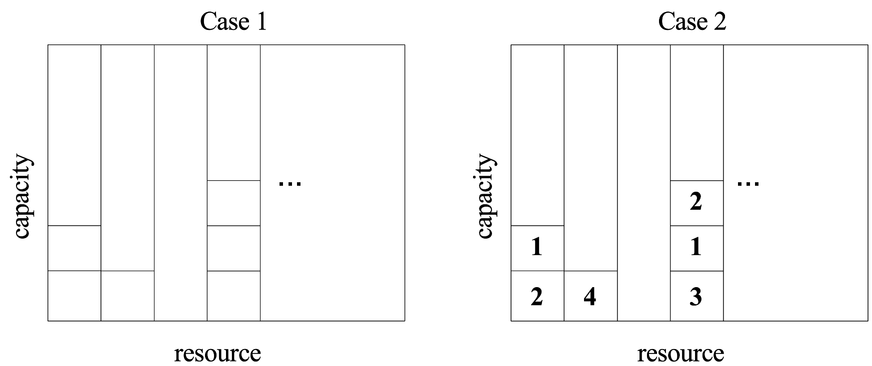


**Figure 2.1.** States in the cases without cancellation process (Case 1) and with cancellation process (Case 2).

In RM, the departure process commonly depends on a customer class. (See p.500 in [38].) It is intuitive that the departure process depends on the customer class which implies size of party in RRM. Actually, the researches in Kimes et al. [22] and Bell and Pliner [3] showed that a correlation between the size of a party and meal duration is significantly positive for real restaurants. Thompson [40] adopted this fact as an assumption to his simulation study.

The state of Case 1 in Figure 2.1 does not need to preserve the customer classes which have arrived until a certain period because of an assumption of the no-departure process. Hence, the state in Case 1 is shown as a vector for capacity. In contrast, the state in Case 2 needs to preserve the customer classes that have arrived until a certain period. This means that each resource in Case 2 have a vector for the customer classes. Thus, the state space of Case 2 is much larger than the one of Case 1. Additionally, if we consider meal duration for each customer which is stated in Kimes et al. [19, 23], Guerriero et al. [14] and etc., then information about how long each customer has been in his/her seat must be added to the state and solving the seating problem as an exact dynamic programming approach is practically impossible.

To broach this argument, we present an exact dynamic programming model with some conditions for seating policies in this chapter.

## 2.2 Formulation

Consider a facility with tables such as a restaurant, cafeteria, etc. Parties arrive to the facility to be served. If they are served there, they depart. The facility has several tables with various kinds of size which means the number of chairs in a table. A system of the facility decides whether to accept arriving parties or not. If the system accepts an arriving party, the system also decides which table to allocate. The system cannot allocate a table to the party of which size is larger than one of the table. The allocated party gives a benefit to the facility. The facility has opening hours. The system aims to maximize the total revenue over the horizon by controlling for arriving parties. These conditions are shown in Figure 2.2.

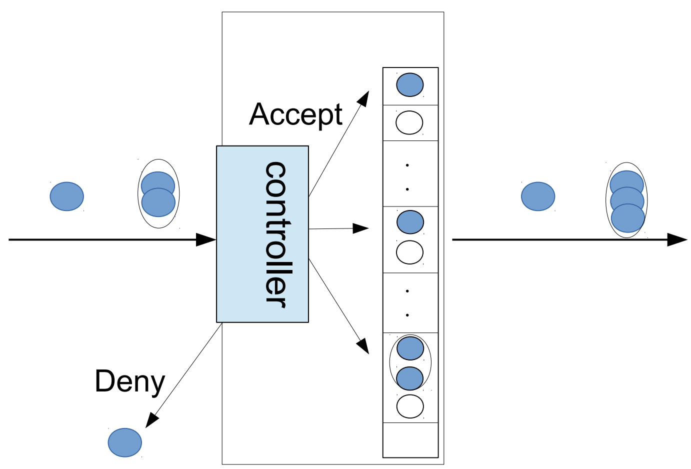


**Figure 2.2.** A facility with arriving parties, tables and decisions.

To simplify a model, some conditions are given to parties and tables. The conditions are that a composition of the tables cannot be modified to suit the arriving parties, sizes of the parties cannot be divided to suit the tables, and the sizes of the parties do not exceed a largest table size in the facility. Further, tables of the same size and seats are not distinguished. Set $P = \{1, \cdots, \overline{P}\}$ and $I = \{1, \cdots, \overline{I}\}$. The notations about the party and the table are shown as

- $\overline{P}$: the number of different party sizes,
- $\overline{I}$: the number of different table sizes,

- $g_p$: the party size for $p \in P$,
- $t_i$: the table size for $i \in I$,
- $m_i$: the number of the table for $i \in I$.

To simplify, we regard $p \in P$ as a party with party size $g_p$, and $i \in I$ as a table with table size $t_i$, respectively. Throughout this chapter, assume that $g_1 < g_2 < \cdots < g_{\overline{P}}$ and $t_1 < t_2 < \cdots < t_{\overline{I}}$. In addition, it is indicated as

- $P_i = \{p \in P : g_p \leq t_i\}$: the party set which is able to be allocated a table $i \in I$ ,
- $I_p = \{i \in I : g_p \leq t_i\}$: the table set which is able to be allocated to the party $p \in P$.

For a set $S$, let $|S|$ be the number of elements of $S$. Set $\overline{P}_i = |P_i|$ and $\overline{I}_p = |I_p|$.

The opening horizon is sufficiently divided into the $N$ periods $n \in \{1, \cdots, N\}$. Set $T_0 = \{0, 1, \cdots, N\}$ and $T_1 = \{1, \cdots, N\}$. One event of the customer's arrival or departure occurs in the period $n$. A period $n = 0$ corresponds to closing of the facility. Parties arrive according to time-dependent Poisson process while the facility is opening. Subramanian et al. [36] explained for this discretization method.

All of them are walk-in customers, without reservation. Departure process of the parties depends on not their length of staying time, but the state of facility and the period. Notations about the state space, the arrival and departure rate, and expected revenue are shown as

- $\overline{X}_i = \{\boldsymbol{x}_i = (x_p^i) : x_p^i \geq 0, p \in P_i\,; \sum_p x_p^i \leq m_i\}$ : state space for a table $i \in I$ where $x_p^i$ is the number of parties who are sitting in a table $i \in I_p$,
- $X_n = \{X = (\boldsymbol{x}_1 | \cdots | \boldsymbol{x}_{\overline{I}}) : \boldsymbol{x}_i \in \overline{X}_i, i \in I; \sum_i \sum_p x_p^i \leq N - n\}$ : state space for a facility with a submatrix $\boldsymbol{x_i}$ in a period $n \in T_0$,
- $r_p^n$: the expected revenue for a party $p \in P$ in a period $n \in T_0$,
- $\lambda_p^n(X)$: the arrival rate for a party $p \in P$ and a state $X \in X_n$ in a period $n \in T_0$, where $\lambda_p^n(X) > 0$,
- $q_{ip}^n(X)$: the departure rate for a party $p \in P_i$ where $i \in I$, and a state $X \in X_n$ in a period $n \in T_0$,

- $\lambda_0^n$: a probability of a null event in period $n \in T_0$.

Referring to p.15 in [34], we can obtain a maximum of $|X_n|$ for $n$: $\chi = \max_n\{|X_n|\}$ as

$$\chi = \prod_{i=1}^{\overline{I}} \binom{m_i + \overline{P}_i}{\overline{P}_i}. \tag{2.1}$$

The (2.1) is helpful to roughly estimate size of state space for a facility. From the assumption of the arrival and the departure process in a period $n \in T_0$, the equation

$$\sum_{p=1}^{\overline{P}} \lambda_p^n(X) + \sum_{p\in\overline{P}} \sum_{i\in I_p} q_{ip}^n(X) + \lambda_0^n(X) = 1 \tag{2.2}$$

is obtained.

For each $n \in T_0$, let $U_n(X)$ be the maximum expected revenue from operating over periods $n$ to 0. Firstly, suppose the maximum expected revenue in a general form as follows.

$$\begin{aligned} U_n(X) = \sum_{p=1}^{\overline{P}} \lambda_p^n(X) \left\{ \left( r_p^n - \min_{i\in I_p} \Delta_p^i U_{n-1}(X) \right)^+ + U_{n-1}(X) \right\} \\ + \sum_{p=1}^{\overline{P}} \sum_{i\in I_p} q_{ip}^n(X) U_{n-1}(X - \boldsymbol{e}_p^i) \\ + \left( 1 - \sum_{p=1}^{\overline{P}} \lambda_p^n(X) - \sum_{p=1}^{\overline{P}} \sum_{i\in I_p} q_{ip}^n(X) \right) U_{n-1}(X), \\ X \in X_n, n \in T_1, \end{aligned} \tag{2.3}$$

where $\boldsymbol{e}_p^i = (\boldsymbol{x}_1|\cdots|\boldsymbol{x}_{\overline{I}})$ in which $x_p^i = 1$ and otherwise $0$, $(a)^+ = \max\{a, 0\}$, and $\Delta_p^i U_n(X) = U_n(X) - U_n(X + \boldsymbol{e}_p^i)$. Boundary conditions are that $U_n(X) = -\infty$ for $X \notin X_n, n \in T_0$, and $U_0(X) = 0$ for $X \in X_0$. Let $n \in T_1$. The $\min_{i\in I_p} \Delta_p^i U_n(X)$ means a threshold price for a party $p \in P_i$ , such that the party $p$ who arrives for the state $X$ in $n$ is acceptable if $r_p^n$ is not less than the threshold price $\min_{i\in I_p} \Delta_p^i U_n(X)$ and not acceptable if $r_p^n$ is less than the threshold price $\min_{i\in I_p} \Delta_p^i U_n(X)$ (See pp.31-32 in [38].). $\Delta_p^i U_n(X)$ is an opportunity cost of accepting the party $p$ for the table $i \in I_p$ in $n$. Note that $\lambda_0^n(X) = 1 - \sum_{p=1}^{\overline{P}} \lambda_p^n(X) - \sum_{p=1}^{\overline{P}} \sum_{i\in I_p} q_{ip}^n(X)$

from (2.2). The first term of the right hand in (2.3) indicates a expected value in a case where a party arrives at a facility in a period $n$. If a $p$ is accepted in the table $i \in I_p$, then a expected value for the case is $r_p^n - \Delta_p^i U_{n-1}(X)$ in $n$. The second term indicates a expected value in a case where a party sitting in a facility leaves in a period $n$. The third term is for a case where no event occurs in a period $n$. From (2.3), an optimal policy is indicated as below.

**Optimal policy:** an optimal policy for a party $p \in P$ and a state $X \in X_n$ is that if $r_p^n - \min_{i \in I_p} \Delta_p^i U_{n-1}(X) \geq 0$, then a party $p$ is accepted in a table $\arg\min_{i \in I_p} \Delta_p^i U_{n-1}(X)$, and if $r_p^n - \min_{i \in I_p} \Delta_p^i U_{n-1}(X) < 0$, then a party $p$ is denied.

Then, some assumptions are supposed to simplify (2.3).

**Assumption 2.1.** *$\lambda_p^n(X) = \lambda_p^n$ for $p \in P$, $n \in T_0$ and $X \in X_n$.*

**Assumption 2.2.** *$q_{ip}^n(X) = x_p^i q_{ip}^n$ for $i \in I$, $p \in P_i$, $n \in T_0$ and $X \in X_n$.*

Assumption 2.1 indicates that arrival rates do not depend on states, which means that congestion level of a facility does not affect the arrival rates.

Assumption 2.2 indicates that a party $p$ in a table $i$ and a period $n$ departs independently of other parties sitting in other tables, which implies that parties do not memorize how long they are sitting in a facility.

Under these assumptions, (2.3) can be rewritten as

$$\begin{aligned} U_n(X) = \sum_{p=1}^{\overline{P}} \lambda_p^n \left\{ \left( r_p^n - \min_{i \in I_p} \Delta_p^i U_{n-1}(X) \right)^+ + U_{n-1}(X) \right\} \\ + \sum_{p=1}^{\overline{P}} \sum_{i \in I_p} x_p^i q_{ip}^n U_{n-1}(X - \boldsymbol{e}_p^i) \\ + \left( 1 - \sum_{p=1}^{\overline{P}} \lambda_p^n - \sum_{p=1}^{\overline{P}} \sum_{i \in I_p} x_p^i q_{ip}^n \right) U_{n-1}(X), \\ X \in X_n, n \in T_1. \end{aligned} \tag{2.4}$$

Boundary conditions are that $U_n(X) = -\infty$ for $X \notin X_n, n \in T_0$, and $U_0(X) = 0$ for $X \in X_0$. The first term, the second term, and the third term of (2.4) are called the *arrival part*, the *departure* *part*, and the *null part*, respectively. (2.4) is close to a equation which is extended by cancellation process for the model with upgrades which is suggested as (1) in Steinhardt and Gönsch [35]. However, state space of the model in Steinhardt and Gönsch [35] is different from the one which is defined in this paper as previously shown in Figure 2.1. Note that the first term of (2.4) is a case of the one of (1) in Steinhardt and Gönsch [35] because of physical bundles between parties and tables, and the condition on which composition of the tables and size of the parties are fixed. For proofs as following sections, policy vector $\boldsymbol{d}$ is defining.

For $n \in T_1$ and $X \in X_n$, a policy vector $d = (d_p)$ is said to be acceptable if for any $p \in P$, the following conditions (i) or (ii) holds.

(i) $d_p = 0$.

(ii) $d_p \in I_p$ and $X + \boldsymbol{e}_p^{d_p} \in X_{n-1}$.

$D_n(X)$ is defined as

$$D_n(X) = \{\boldsymbol{d} = (d_p) : \boldsymbol{d} \text{ is acceptable}\}, n \in T_1, X \in X_n.$$

## 2.3 Structural Properties

Supposing the Assumption 2.3 as below, a monotonicity which is similar to the monotonicity suggested as Proposition 1 in Steinhardt and Gönsch [35] is obtained for $\Delta_p^i U_n(X)$ in (2.4).

**Assumption 2.3.** *$q_{\delta p}^n = q_{\delta' p}^n$ for $p \in P$ with $\overline{I}_p \geq 2$, $\delta, \delta' \in I_p$ with $\delta \neq \delta'$, and $n \in T_0$.*

**Lemma 2.4.** *Under Assumptions 2.1 to 2.3, for a given $p \in P$, $n \in T_0$ and $X \in X_n$,*

$$\Delta_p^{\delta} U_n(X) \leq \Delta_p^{\delta'} U_n(X) \tag{2.5}$$

*where $\delta$, $\delta' \in I_p$, $t_\delta < t_{\delta'}$, $\sum_p x_p^{\delta} < m_\delta$, and $\sum_p x_p^{\delta'} < m_{\delta'}$.*

***Proof:*** $U_n(X + \boldsymbol{e}_p^{\delta}) \geq U_n(X + \boldsymbol{e}_p^{\delta'})$ should be indicated by induction for $\Delta_p^{\delta} U_n(X) \leq \Delta_p^{\delta'} U_n(X)$. For $n = 0$, it is obvious that $U_0(X + \boldsymbol{e}_p^{\delta}) = U_0(X + \boldsymbol{e}_p^{\delta'}) = 0$. Then, assume that $U_{n-1}(X + \boldsymbol{e}_p^{\delta}) \geq U_{n-1}(X + \boldsymbol{e}_p^{\delta'})$. In the following, we are indicating the orderings of each part.

Firstly, an order of the arrival part is indicated. The arrival part of (2.4) is rewritten using the optimal vector as

$$\max_{\boldsymbol{d}\in D_n(X)}\left\{\sum_{p|d_p\neq 0}\lambda_p^n(r_p^n+U_{n-1}(X+\boldsymbol{e}_p^{d_p}))+\sum_{p|d_p=0}\lambda_p^n U_{n-1}(X)\right\}.$$

Let optimal policy vectors for $U_n(X+\boldsymbol{e}_p^{\delta})$ and $U_n(X+\boldsymbol{e}_p^{\delta'})$ be $\boldsymbol{d}^{(\delta)*}$ and $\boldsymbol{d}^{(\delta')*}$, respectively.

For a given $p\in P$, there are four cases for $d_p^{(\delta)*}$ and $d_p^{(\delta')*}$ as follows.

i) In the case $d_p^{(\delta)*}\neq 0$ and $d_p^{(\delta')*}\neq 0$, we should make a comparison between $r_p^n+U_{n-1}(X+\boldsymbol{e}_p^{\delta}+\boldsymbol{e}_p^{d_p^{(\delta)*}})$ and $r_p^n+U_{n-1}(X+\boldsymbol{e}_p^{\delta'}+\boldsymbol{e}_p^{d_p^{(\delta')*}})$ for the arrival parts of $U_n(X+\boldsymbol{e}_p^{\delta})$ and $U_n(X+\boldsymbol{e}_p^{\delta'})$. Further, this case is divided into two cases for ordering between $d_p^{(\delta)*}$ and $d_p^{(\delta')*}$.

i-1) In the case $d_p^{(\delta)*}\leq d_p^{(\delta')*}$, from the inductive hypothesis, $r_p^n+U_{n-1}(X+\boldsymbol{e}_p^{\delta}+\boldsymbol{e}_p^{d_p^{(\delta)*}})\geq r_p^n+U_{n-1}(X+\boldsymbol{e}_p^{\delta'}+\boldsymbol{e}_p^{d_p^{(\delta)*}})\geq r_p^n+U_{n-1}(X+\boldsymbol{e}_p^{\delta'}+\boldsymbol{e}_p^{d_p^{(\delta')*}})$ is obtained

i-2) In the case $d_p^{(\delta)*}>d_p^{(\delta')*}$, from the inductive hypothesis and number of capacities of tables, $d_p^{(\delta)*}\leq\delta'$ and $d_p^{(\delta')*}=\delta$ is obtained. Thus, $r_p^n+U_{n-1}(X+\boldsymbol{e}_p^{\delta}+\boldsymbol{e}_p^{d_p^{(\delta)*}})\geq r_p^n+U_{n-1}(X+\boldsymbol{e}_p^{\delta'}+\boldsymbol{e}_p^{\delta})=r_p^n+U_{n-1}(X+\boldsymbol{e}_p^{\delta'}+\boldsymbol{e}_p^{d_p^{(\delta')*}})$.

ii) In the case $d_p^{(\delta)*}=0$ and $d_p^{(\delta')*}\neq 0$, we should make a comparison between $U_{n-1}(X+\boldsymbol{e}_p^{\delta})$ and $r_p^n+U_{n-1}(X+\boldsymbol{e}_p^{\delta'}+\boldsymbol{e}_p^{d_p^{(\delta')*}})$. From the inductive hypothesis and $d_p^{(\delta)*}=0$, $U_{n-1}(X+\boldsymbol{e}_p^{\delta})\geq r_p^n+U_{n-1}(X+\boldsymbol{e}_p^{\delta}+\boldsymbol{e}_p^{d_p^{(\delta')*}})\geq r_p^n+U_{n-1}(X+\boldsymbol{e}_p^{\delta'}+\boldsymbol{e}_p^{d_p^{(\delta')*}})$.

iii) In the case $d_p^{(\delta)*}\neq 0$ and $d_p^{(\delta')*}=0$, we should make a comparison between $r_p^n+U_{n-1}(X+\boldsymbol{e}_p^{\delta}+\boldsymbol{e}_p^{d_p^{(\delta)*}})$ and $U_{n-1}(X+\boldsymbol{e}_p^{\delta'})$. From the inductive hypothesis and $d_p^{(\delta)*}\neq 0$, $r_p^n+U_{n-1}(X+\boldsymbol{e}_p^{\delta}+\boldsymbol{e}_p^{d_p^{(\delta)*}})\geq U_{n-1}(X+\boldsymbol{e}_p^{\delta})\geq U_{n-1}(X+\boldsymbol{e}_p^{\delta'})$.

iv) In the case $d_p^{(\delta)*}=d_p^{(\delta')*}=0$, from the inductive hypothesis, It is obvious that $U_{n-1}(X+\boldsymbol{e}_p^{\delta})\geq U_{n-1}(X+\boldsymbol{e}_p^{\delta'})$.

Next, we consider the departure parts. To simplify the notation, we set $q_p^n=q_{ip}^n, i\in I_p$. For the $p$, the departure parts of $U_n(X+\boldsymbol{e}_p^{\delta})$ and $U_n(X+\boldsymbol{e}_p^{\delta'})$ are

$$\sum_{i\in I_p}(x_p^i+e_p^{\delta i})q_p^n U_{n-1}(X+\boldsymbol{e}_p^{\delta}-\boldsymbol{e}_p^i)\tag{2.6}$$

and

$$\sum_{i \in I_p} (x_p^i + e_p^{\delta' i}) q_p^n U_{n-1}(X + \boldsymbol{e}_p^{\delta'} - \boldsymbol{e}_p^i), \tag{2.7}$$

respectively, where $e_p^{ki} = 1$ if $i = k$ and otherwise $e_p^{ki} = 0$. (2.6) and (2.7) can stand for

$$\begin{aligned} q_p^n \left\{ \cdots + (x_p^\delta + 1) U_{n-1}(X + \boldsymbol{e}_p^\delta - \boldsymbol{e}_p^\delta) + \cdots + x_p^{\delta'} U_{n-1}(X + \boldsymbol{e}_p^\delta - \boldsymbol{e}_p^{\delta'}) + \cdots \right\} \\ = q_p^n \left\{ U_{n-1}(X) + x_p^1 U_{n-1}(X + \boldsymbol{e}_p^\delta - \boldsymbol{e}_p^1) + \cdots \right\} \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} q_p^n \left\{ \cdots + x_p^\delta U_{n-1}(X + \boldsymbol{e}_p^{\delta'} - \boldsymbol{e}_p^\delta) + \cdots + (x_p^{\delta'} + 1) U_{n-1}(X + \boldsymbol{e}_p^{\delta'} - \boldsymbol{e}_p^{\delta'}) + \cdots \right\} \\ = q_p^n \left\{ U_{n-1}(X) + x_p^1 U_{n-1}(X + \boldsymbol{e}_p^{\delta'} - \boldsymbol{e}_p^1) + \cdots \right\}, \end{aligned} \tag{2.9}$$

respectively. Therefore, from the inductive hypothesis,

$$\sum_{i \in I_p} (x_p^i + e_p^{\delta i}) q_p^n U_{n-1}(X + \boldsymbol{e}_p^\delta - \boldsymbol{e}_p^i) \geq \sum_{i \in I_p} (x_p^i + e_p^{\delta' i}) q_p^n U_{n-1}(X + \boldsymbol{e}_p^{\delta'} - \boldsymbol{e}_p^i)$$

is obtained.

Finally, we consider the null parts. For the $p$, the null parts of $U_n(X + \boldsymbol{e}_p^\delta)$ and $U_n(X + \boldsymbol{e}_p^{\delta'})$ are

$$\left( 1 - \lambda_p^n - \sum_{i \in I_p} (x_p^i + e_p^{\delta i}) q_p^n \right) U_{n-1}(X + \boldsymbol{e}_p^\delta) \tag{2.10}$$

and

$$\left( 1 - \lambda_p^n - \sum_{i \in I_p} (x_p^i + e_p^{\delta' i}) q_p^n \right) U_{n-1}(X + \boldsymbol{e}_p^{\delta'}), \tag{2.11}$$

respectively. In these equations, the coefficients of the $U_{n-1}(X + \boldsymbol{e}_p^\delta)$ and $U_{n-1}(X + \boldsymbol{e}_p^{\delta'})$ are the same. Thus,

$$\left(1-\lambda_p^n-\sum_{i\in I_p}(x_p^i+e_p^\delta)q_p^n\right)U_{n-1}(X+\boldsymbol{e}_p^\delta)\geq\left(1-\lambda_p^n-\sum_{i\in I_p}(x_p^i+e_p^{\delta'})q_p^n\right)U_{n-1}(X+\boldsymbol{e}_p^{\delta'})$$

is obtained from the inductive hypothesis.

From these ordering of the arrival parts, the departure parts, and the null parts of $U_n(X+\boldsymbol{e}_p^\delta)$ and $U_n(X+\boldsymbol{e}_p^{\delta'})$, the (2.5) is indicated. ■

Assumption 2.3 means that departure rate depends on only a period and a party size. For $p \in P$ and $n \in T_0$, we denote $q_{ip}^n, i \in I_p$ as $q_p^n$. For this assumption, Kimes et al. [20] suggested that meal duration which relates to the departure rate did not depend on position, configuration, and size of tables while it depended on the size of a party. Robson and Kimes [31] reported that meal duration of a party which sits in an oversize table is not different from the one of a party which sits in a right-size table. Therefore, Assumption 2.3 can be considered as realistic one.

For $n \in T_0, i \in I$ and the submatrix $\boldsymbol{x}_i$ of $X \in X_n$, set $x^i := \sum_p x_p^i$. Furthermore, let $X \in X_n$ and $\hat{X} \in X_n$ be the states with submatrices $\boldsymbol{x}_i$ and $\hat{\boldsymbol{x}}_i$, respectively, where $X \neq \hat{X}$ and $x^i = \hat{x}^i$ for $i \in I$. This assumption for $X$ and $\hat{X}$ is used in the remainder of this section.

Claim 2.5 is obtained from Lemma 2.4.

**Claim 2.5.** *Let $\boldsymbol{d}^* = (d_p^*)$ and $\hat{\boldsymbol{d}}^* = (\hat{d}_p^*)$ be optimal policy vectors for the $X$ and $\hat{X}$, respectively. For each $p \in P$, if $d_p^* \neq 0$ and $\hat{d}_p^* \neq 0$, then $d_p^* = \hat{d}_p^*$.*

***Proof:*** From $d_p^* \neq 0$ and $\hat{d}_p^* \neq 0$, arrival parts of $U_n(X)$ and $U_n(\hat{X})$ are

$$\lambda_p^n(r_p^n + U_{n-1}(X+\boldsymbol{e}^{d_p^*})) \tag{2.12}$$

and

$$\lambda_p^n(r_p^n + U_{n-1}(\hat{X}+\boldsymbol{e}^{\hat{d}_p^*})), \tag{2.13}$$

respectively. From $x^i = \hat{x}^i$, the table sets which are able to be $d_p^*$ and $\hat{d}_p^*$ for $p \in P$ are the same. Then, $d_p^* = \hat{d}_p^*$ is obtained. ■

Suppose an assumption for the ordering of departure process of parties $p \in P$, and a proposition about a monotonicity of $\Delta_p^i U_n(X)$ for $p \in P$ as below.

**Assumption 2.6.** *For $n \in T_0$ and $\psi, \psi' \in P$ with $\psi < \psi'$, $q_\psi^n \geq q_{\psi'}^n$.*

**Proposition 2.7.** *Under Assumptions 2.1 to 2.6, for a given $\delta \in I$ with $\overline{P}_\delta \geq 2$, $\psi, \psi' \in P_\delta$ with $\psi < \psi'$ and $n \in T_0$,*

$$\Delta_\psi^\delta U_n(X) \leq \Delta_{\psi'}^\delta U_n(X). \tag{2.14}$$

***Proof:*** It is obtained by induction. $U_n(X + \boldsymbol{e}_\psi^\delta) \geq U_n(X + \boldsymbol{e}_{\psi'}^\delta)$ should be indicated for $\Delta_\psi^\delta U_n(X) \leq \Delta_{\psi'}^\delta U_n(X)$. In the case $n = 0$, $U_0(X + \boldsymbol{e}_\psi^\delta) = U_0(X + \boldsymbol{e}_{\psi'}^\delta)$ is clear. Then, assume that $\Delta_\psi^\delta U_{n-1}(X) \leq \Delta_{\psi'}^\delta U_{n-1}(X)$.

Firstly, we consider about the arrival parts. Let the optimal vectors for the states $X + \boldsymbol{e}_\psi^\delta$ and $X + \boldsymbol{e}_{\psi'}^\delta$ be $\boldsymbol{d}^{(\psi)^*}$ and $\boldsymbol{d}^{(\psi')^*}$, respectively.

i) In the case $d_p^{(\psi)^*} \neq 0$ and $d_p^{(\psi')^*} \neq 0$, we make a comparison between $r_p^n + U_{n-1}(X + \boldsymbol{e}_\psi^\delta + \boldsymbol{e}_p^{d_p^{(\psi)^*}})$ and $r_p^n + U_{n-1}(X + \boldsymbol{e}_{\psi'}^\delta + \boldsymbol{e}_p^{d_p^{(\psi')^*}})$. The optimal vectors for the states $X + \boldsymbol{e}_\psi^\delta$ and $X + \boldsymbol{e}_{\psi'}^\delta$ are $d_p^{(\psi)^*} = d_p^{(\psi')^*}$ from Claim 1 because capacities of the states are the same. Hence, $r_p^n + U_{n-1}(X + \boldsymbol{e}_\psi^\delta + \boldsymbol{e}_p^{d_p^{(\psi)^*}}) \geq r_p^n + U_{n-1}(X + \boldsymbol{e}_{\psi'}^\delta + \boldsymbol{e}_p^{d_p^{(\psi')^*}})$ is indicated.

ii) In the case $d_p^{(\psi)^*} \neq 0$ and $d_p^{(\psi')^*} = 0$, we compare $r_p^n + U_{n-1}(X + \boldsymbol{e}_\psi^\delta + \boldsymbol{e}_p^{d_p^{(\psi)^*}})$ to $U_{n-1}(X + \boldsymbol{e}_{\psi'}^\delta)$. From the inductive hypothesis and $d_p^{(\psi')^*} = 0$, $r_p^n + U_{n-1}(X + \boldsymbol{e}_\psi^\delta + \boldsymbol{e}_p^{d_p^{(\psi)^*}}) \geq U_{n-1}(X + \boldsymbol{e}_\psi^\delta) \geq U_{n-1}(X + \boldsymbol{e}_{\psi'}^\delta)$ is obtained.

iii) In the case $d_p^{(\psi)^*} = 0$ and $d_p^{(\psi')^*} \neq 0$, we make a comparison between $U_{n-1}(X + \boldsymbol{e}_\psi^\delta)$ and $r_p^n + U_{n-1}(X + \boldsymbol{e}_{\psi'}^\delta + \boldsymbol{e}_p^{d_p^{(\psi')^*}})$. From the inductive hypothesis and $d_p^{(\psi)^*} = 0$, $U_{n-1}(X + \boldsymbol{e}_\psi^\delta) > r_p^n + U_{n-1}(X + \boldsymbol{e}_\psi^\delta + \boldsymbol{e}_p^{d_p^{(\psi)^*}}) \geq r_p^n + U_{n-1}(X + \boldsymbol{e}_\psi^\delta + \boldsymbol{e}_p^{d_p^{(\psi')^*}}) \geq r_p^n + U_{n-1}(X + \boldsymbol{e}_{\psi'}^\delta + \boldsymbol{e}_p^{d_p^{(\psi')^*}})$ is obtained.

iv) In the case $d_p^{(\psi)^*} = 0$ and $d_p^{(\psi')^*} = 0$, it is obvious.

Then, we consider the departure parts of $U_n(X + \boldsymbol{e}_\psi^\delta)$ and $U_n(X + \boldsymbol{e}_{\psi'}^\delta)$ which are

$$\sum_{p=1}^{\overline{P}} \sum_{i \in I_p} (x_p^i + e_{\psi p}^{\delta i}) q_p^n U_{n-1}(X + \boldsymbol{e}_p^\delta - \boldsymbol{e}_p^i) \tag{2.15}$$

and

$$\sum_{p=1}^{\overline{P}} \sum_{i \in I_p} (x_p^i + e_{\psi' p}^{\delta i}) q_p^n U_{n-1}(X + \boldsymbol{e}_p^{\delta'} - \boldsymbol{e}_p^i), \tag{2.16}$$

respectively, where $e_{lp}^{ki} = 1$ if $i = k$ and $p = l$, otherwise $e_{lp}^{ki} = 0$.

We should consider only the cases $p = \psi$, $i = \sigma$ and $p = \psi'$, $i = \delta$ for (2.15) and (2.16) as

$$\begin{aligned} \cdots + q_\psi^n U_{n-1}(X) + x_\psi^\delta q_\psi^n U_{n-1}(X + \boldsymbol{e}_\psi^\delta - \boldsymbol{e}_\psi^\delta) + \cdots \\ \cdots + x_{\psi'}^\delta q_{\psi'}^n U_{n-1}(X + \boldsymbol{e}_\psi^\delta - \boldsymbol{e}_{\psi'}^\delta) + \cdots \end{aligned} \tag{2.17}$$

and

$$\begin{aligned} \cdots + x_\psi^\delta q_\psi^n U_{n-1}(X + \boldsymbol{e}_{\psi'}^\delta - \boldsymbol{e}_\psi^\delta) + \cdots \\ \cdots + q_{\psi'}^n U_{n-1}(X) + x_{\psi'}^\delta q_{\psi'}^n U_{n-1}(X + \boldsymbol{e}_{\psi'}^\delta - \boldsymbol{e}_{\psi'}^\delta) + \cdots . \end{aligned} \tag{2.18}$$

From the inductive hypothesis and Assumption 2.6, it is indicated that

$$\sum_{p=1}^{\overline{P}} \sum_{i \in I_p} (x_p^i + e_{\psi p}^{\delta i}) q_p^n U_{n-1}(X + \boldsymbol{e}_p^\delta - \boldsymbol{e}_p^i) \geq \sum_{p=1}^{\overline{P}} \sum_{i \in I_p} (x_p^i + e_{\psi' p}^{\delta i}) q_p^n U_{n-1}(X + \boldsymbol{e}_p^{\delta'} - \boldsymbol{e}_p^i).$$

Finally, we consider the null parts. It is clear that coefficients of the null parts of $U_n(X + \boldsymbol{e}_\psi^\delta)$ and $U_n(X + \boldsymbol{e}_{\psi'}^\delta)$ are the same.

From the ordering of the each part, we obtain that $\Delta_\psi^\delta U_n(X) \leq \Delta_{\psi'}^\delta U_n(X)$. ■

Assumption 2.6 means that a party stochastically stays longer than the smaller one. Assumption 2.6 is considered as realistic one as previously mentioned in section 2.1. Remark 2.8 for Proposition 2.7 is indicated as follows.

**Remark 2.8.** Note that the monotonicity of Proposition 2.7 does not depend on the expected revenue $r_p^n$, which is same as Lemma 2.4. Seeing the proof for Proposition 2.7, we can recognize that Assumption 2.6 is used only in the terms of $U_{n-1}(X)$ in (2.17) and (2.18). Further, the orderings for the each part expect the the terms of $U_{n-1}(X)$ in (2.17) and (2.18) is conditioned by the inductive hypothesis and facts of the cases. Thus, the ordering of Proposition 2.7 is conditioned by only the ordering of departure rates between the parties.

Thus, from Proposition 2.7 and its character, a difference between the maximal expected revenues $U_n(X)$ and $U_n(\hat{X})$ stems from differences for departure rates among parties. If there are differences for departure rates among parties, then they are affected by all factors; arrival rates, rewards, and etc. as a matter of course. However, if there are not the differences for departure rates among parties, then there is not the difference between $U_n(X)$ and $U_n(\hat{X})$, nevertheless the parties have difference parameters each other.

From the monotonicities which is indicated in this paper, a sufficient condition which is able to reduce variations of optimal policies can be obtained. The sufficient condition is shown as Theorem 2.9. For a given party $p \in P$, set $\overline{d}_p^* = \min\{i \in I_p : m_i - \sum_{k \in P_i} x_k^i > 0\}$.

**Theorem 2.9.** *If the condition*

$$r_p^n \notin \left[\min\{\Delta_p^{\overline{d}_p^*} U_{n-1}(X), \Delta_p^{\overline{d}_p^*} U_{n-1}(\hat{X})\}, \max\{\Delta_p^{\overline{d}_p^*} U_{n-1}(X), \Delta_p^{\overline{d}_p^*} U_{n-1}(\hat{X})\}\right] \tag{2.19}$$

*is satisfied for a given $p \in P$ and $n \in T_1$, then $d_p^* = \hat{d}_p^*$, where $\boldsymbol{d}^* = (d_p^*)$ and $\hat{\boldsymbol{d}}^* = (\hat{d}_p^*)$ are the optimal vectors for the states $X$ and $\hat{X}$, respectively.*

***Proof:*** i) In the case $d_p^* \neq 0$ and $\hat{d}_p^* \neq 0$, from Claim 2.5, $d_p^* = \hat{d}_p^*$.
ii) In the case $d_p^* = 0$ and $\hat{d}_p^* \neq 0$, from $d_p^* = 0$, we obtain that

$$\lambda_p^n U_{n-1}(X) > \max_{d_p | d_p \neq 0} \left\{ \lambda_p^n (r_p^n + U_{n-1}(X + \boldsymbol{e}_p^{d_p})) \right\}. \tag{2.20}$$

In addition, (2.20) can be rewritten to

$$\lambda_p^n U_{n-1}(X) > \lambda_p^n (r_p^n + U_{n-1}(X + \boldsymbol{e}_p^{\hat{d}_p^*})) \tag{2.21}$$

from the condition $x^i = \hat{x}^i$. We also obtain that

$$\lambda_p^n U_{n-1}(\hat{X}) \leq \lambda_p^n (r_p^n + U_{n-1}(\hat{X} + \boldsymbol{e}_p^{\hat{d}_p^*})) \tag{2.22}$$

because of $\hat{d}_p^* \neq 0$. From (2.21) and (2.22), we indicate

$$\Delta_p^{\hat{d}_p^*} U_{n-1}(\hat{X}) \leq r_p^n < \Delta_p^{\hat{d}_p^*} U_{n-1}(X) \tag{2.23}$$

as a condition for $d_p^* = 0$ and $\hat{d}_p^* \neq 0$.

iii) In the case $d_p^* \neq 0$ and $\hat{d}_p^* = 0$, calculating this case similar to the case ii), we can obtain

$$\Delta_p^{d_p^*} U_{n-1}(X) \leq r_p^n < \Delta_p^{d_p^*} U_{n-1}(\hat{X}) \tag{2.24}$$

as a condition for $d_p^* \neq 0$ and $\hat{d}_p^* = 0$.

iv) In the case $d_p^* = 0$ and $\hat{d}_p^* = 0$, it is clear.

Then, the relation between $\hat{d}_p^*$ and $d_p^*$ in (2.23) and (2.24) is $\hat{d}_p^* = d_p^* = \overline{d}_p^*$ due to $x^i = \hat{x}^i$, Lemma 2.4, and $\hat{d}_p^*, d_p^* \neq 0$. Therefore, if a range which does not include the ranges (2.23) and (2.24):

$$r_p^n \notin \left[ \min\{\Delta_p^{\overline{d}_p^*} U_{n-1}(X), \Delta_p^{\overline{d}_p^*} U_{n-1}(\hat{X})\}, \max\{\Delta_p^{\overline{d}_p^*} U_{n-1}(X), \Delta_p^{\overline{d}_p^*} U_{n-1}(\hat{X})\} \right] \tag{2.25}$$

is satisfied, then $d_p^* = \hat{d}_p^*$. ■

The remark of the Theorem 2.9 is below.

**Remark 2.10.** The range (2.19) indicates a sufficient condition which makes the same optimal policy for the state $X$ and $\hat{X}$. The width of the range $|\Delta_p^{\overline{d}_p^*} U_{n-1}(X) - \Delta_p^{\overline{d}_p^*} U_{n-1}(\hat{X})|$ stands for difficulty of reducing variety of the optimal policies. If the width becomes narrower, then it is more difficult to insert the expected revenue $r_p^n$ into the range and optimal policies go to depend on only capacities for tables.

The width of the range $|\Delta_p^{\overline{d}_p^*} U_{n-1}(X) - \Delta_p^{\overline{d}_p^*} U_{n-1}(\hat{X})|$ can be rewritten to $|U_{n-1}(\hat{X}) - U_{n-1}(X) + U_{n-1}(X + \boldsymbol{e}_p^{\overline{d}_p^*}) - U_{n-1}(\hat{X} + \boldsymbol{e}_p^{\overline{d}_p^*})|$ where Proposition 2.7 is applicable to $U_{n-1}(\hat{X}) - U_{n-1}(X)$ and $U_{n-1}(X + \boldsymbol{e}_p^{\overline{d}_p^*}) - U_{n-1}(\hat{X} + \boldsymbol{e}_p^{\overline{d}_p^*})$. If there are not differences in departure rates among parties, then the width is effected by nothing because the width is zero, regardless of existing differences in arrival rates or expected revenues among the parties. As a consequence of this property, existing the differences in departure rates among parties is an only trigger for expanding varieties of optimal policies.

## 2.4 Numerical Examples

In this section, we confirm the feature which is stated in Remark 2.10. Numerical examples are computed using an equation which is applied Assumptions 2.1 to 2.6 to (2.4). Configurations for tables and parties are which $N = 20, \overline{P} = 2,\ \overline{I} = 2,\ g_1 = 1, g_2 = 2,\ t_1 = 1, t_2 = 2,\ m_1 = 2,$ and $m_2 = 2$. From this parameter set, $\chi = 18$ by using (2.1). Arrival rates, departure rates, and expected revenues for each party $p \in P$ and $n \in T_0$ are shown in Table 2.1.

The parameters set in Table 2.1 is named Sample 1. Sample 1 has a single peak for the arrival rates, departure rates, and expected revenues. The expected revenues in Sample 1 are set to increase as they get closer to the peak time. Optimal policies for $p = 1$ which is computed from Sample 1 are shown in Table 2.2. The values in cells of Table 2.2 stand for policy vectors.

Seeing optimal policies for states (2|1,0) and (2|0,1), we can find that the optimal policies in $n = 16$ and 17 are difference between the states; nevertheless capacities for the states are the same. Let the states (2|1,0) and (2|0,1) be $X$ and $\hat{X}$, respectively. To confirm Theorem 2.9 for the states, $\Delta_1^{\overline{d}_1^*} U_{n-1}(X)$ and $\Delta_1^{\overline{d}_1^*} U_{n-1}(\hat{X})$ where $\overline{d}_1^* = 2$, are shown in Table 2.3 which also includes the expected revenue $r_1^n$ to make a comparison easily.

**Table 2.1.** Arrival rates, departure rates, and expected revenues of Sample 1.

| | Arrival Rate | | Departure Rate | | Reward | |
|---|---|---|---|---|---|---|
| $n$ | $\lambda_1^n$ | $\lambda_2^n$ | $q_1^n$ | $q_2^n$ | $r_1^n$ | $r_2^n$ |
| 0-5 | .021 | .014 | .018 | .014 | 3 | 6 |
| 6-7 | .105 | .070 | .088 | .070 | 4 | 8 |
| 8-11 | .150 | .100 | .125 | .100 | 5 | 10 |
| 12-13 | .105 | .070 | .088 | .070 | 4 | 8 |
| 14-20 | .021 | .014 | .018 | .014 | 3 | 6 |

$r_1^{16}$ and $r_1^{17}$ are put in the ranges between $\Delta_1^{\overline{d}_1^*} U_{n-1}(X)$ and $\Delta_1^{\overline{d}_1^*} U_{n-1}(\hat{X})$ for $n = 16$ and $n = 17$. Then, for a case where there is not difference in departure rates between parties, we have computed the range. Sample 2 is the case in which the departure rates for $p = 2$ become the same as the ones for $p = 1$ for Sample 1. $\Delta_1^{\overline{d}_1^*} U_{n-1}(X)$, $\Delta_1^{\overline{d}_1^*} U_{n-1}(\hat{X})$, and the width of the range are shown as Table 2.4.

**Table 2.2.** Optimal policies for $p = 1$ in periods $n$.

| $n\backslash X$ | 2\|0,2 | 2\|2,0 | 2\|0,0 | 2\|1,1 | 2\|0,1 | 2\|1,0 | 0\|0,2 | 0\|2,0 | 0\|0,0 | 0\|1,1 | 0\|0,1 | 0\|1,0 | 1\|0,2 | 1\|2,0 | 1\|0,0 | 1\|1,1 | 1\|0,1 | 1\|1,0 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – |
| 1 | 0 | 0 | 2 | 0 | 2 | 2 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 0 | 0 | 2 | 0 | 2 | 2 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 3 | 0 | 0 | 2 | 0 | 2 | 2 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 4 | 0 | 0 | 2 | 0 | 2 | 2 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 5 | 0 | 0 | 2 | 0 | 2 | 2 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 6 | 0 | 0 | 2 | 0 | 2 | 2 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 7 | 0 | 0 | 2 | 0 | 2 | 2 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 8 | 0 | 0 | 2 | 0 | 2 | 2 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 9 | 0 | 0 | 2 | 0 | 2 | 2 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 10 | 0 | 0 | 2 | 0 | 2 | 2 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 11 | 0 | 0 | 2 | 0 | 2 | 2 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 12 | 0 | 0 | 2 | 0 | 2 | 2 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 13 | 0 | 0 | 2 | 0 | 2 | 2 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 14 | 0 | 0 | 2 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 15 | 0 | 0 | 2 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 16 | 0 | 0 | 2 | 0 | 0 | 2 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 17 | – | – | 2 | – | 0 | 2 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 18 | – | – | 2 | – | – | – | 1 | 1 | 1 | 1 | 1 | 1 | – | – | 1 | – | 1 | 1 |
| 19 | – | – | – | – | – | – | – | – | 1 | – | 1 | 1 | – | – | 1 | – | – | – |
| 20 | – | – | – | – | – | – | – | – | 1 | – | – | – | – | – | – | – | – | – |

**Table 2.3.** The range in Theorem 2.9 for the states $X$ and $\hat{X}$.

| n | $r_1^n$ | $\Delta_1^{\overline{d}_1^*} U_{n-1}(X)$ | $\Delta_1^{\overline{d}_1^*} U_{n-1}(\hat{X})$ |
|---|---|---|---|
| 0 | 3 | – | – |
| 1 | 3 | 0.000 | 0.000 |
| 2 | 3 | 0.147 | 0.147 |
| 3 | 3 | 0.282 | 0.282 |
| 4 | 3 | 0.405 | 0.406 |
| 5 | 3 | 0.518 | 0.521 |
| 6 | 4 | 0.622 | 0.626 |
| 7 | 4 | 1.348 | 1.360 |
| 8 | 5 | 1.785 | 1.814 |
| 9 | 5 | 2.551 | 2.601 |
| 10 | 5 | 2.932 | 3.006 |
| 11 | 5 | 3.174 | 3.262 |
| 12 | 4 | 3.337 | 3.434 |
| 13 | 4 | 3.172 | 3.272 |
| 14 | 3 | 3.095 | 3.193 |
| 15 | 3 | 3.040 | 3.140 |
| 16 | 3 | 2.989 | 3.090 |
| 17 | 3 | 2.941 | 3.043 |

**Table 2.4.** The range and the difference for Sample 2.

| n | $\Delta_1^{\overline{d}_1^*} U_{n-1}(\hat{X})$ | $\Delta_1^{\overline{d}_1^*} U_{n-1}(X)$ | Dif. |
|---|---|---|---|
| 0 | – | – | – |
| 1 | 0.000 | 0.000 | 0.000 |
| 2 | 0.147 | 0.147 | 0.000 |
| 3 | 0.282 | 0.282 | 0.000 |
| 4 | 0.405 | 0.405 | 0.000 |
| 5 | 0.518 | 0.518 | 0.000 |
| 6 | 0.622 | 0.622 | 0.000 |
| 7 | 1.348 | 1.348 | 0.000 |
| 8 | 1.786 | 1.786 | 0.000 |
| 9 | 2.555 | 2.555 | 0.000 |
| 10 | 2.940 | 2.940 | 0.000 |
| 11 | 3.189 | 3.189 | 0.000 |
| 12 | 3.359 | 3.359 | 0.000 |
| 13 | 3.199 | 3.199 | 0.000 |
| 14 | 3.128 | 3.128 | 0.000 |
| 15 | 3.075 | 3.075 | 0.000 |
| 16 | 3.026 | 3.026 | 0.000 |
| 17 | 2.980 | 2.980 | 0.000 |

We can confirm that the width of (2.19) is zero since there is not difference in the departure rates between the parties. Remember that there is difference in the arrival rates and the expected revenues between the parties. Additionally, how the range has influence on the difference for departure rates is indicated. Let additional datasets in where the departure rate for $p = 2$ is multiplied by $0.75$, $0.5$, and $0.25$ for Sample 1 be Sample 3, 4, and 5, respectively. The widths of the ranges for the states $X$ and $\hat{X}$ which are computed from Sample 1 to 5 are shown in Table 2.5. We can recognize that the widths enlarge for all $n$ if the differences for the departure rates enlarge. Thus, what increasing difference for the departure rates enlarges the width of the range is suggested.

**Table 2.5.** The widths of the ranges for the samples.

| n | Sample2 | Sample1 | Sample3 | Sample4 | Sample5 |
|---|---|---|---|---|---|
| 1 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| 2 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| 3 | 0.000 | 0.001 | 0.001 | 0.002 | 0.002 |
| 4 | 0.000 | 0.001 | 0.003 | 0.004 | 0.006 |
| 5 | 0.000 | 0.003 | 0.005 | 0.008 | 0.011 |
| 6 | 0.000 | 0.004 | 0.008 | 0.013 | 0.017 |
| 7 | 0.000 | 0.013 | 0.025 | 0.038 | 0.052 |
| 8 | 0.000 | 0.029 | 0.058 | 0.088 | 0.119 |
| 9 | 0.000 | 0.050 | 0.101 | 0.154 | 0.209 |
| 10 | 0.000 | 0.074 | 0.150 | 0.229 | 0.310 |
| 11 | 0.000 | 0.088 | 0.180 | 0.275 | 0.374 |
| 12 | 0.000 | 0.097 | 0.198 | 0.302 | 0.410 |
| 13 | 0.000 | 0.101 | 0.204 | 0.311 | 0.421 |
| 14 | 0.000 | 0.098 | 0.199 | 0.302 | 0.407 |
| 15 | 0.000 | 0.100 | 0.202 | 0.306 | 0.411 |
| 16 | 0.000 | 0.101 | 0.204 | 0.308 | 0.411 |
| 17 | 0.000 | 0.101 | 0.204 | 0.308 | 0.410 |

## 2.5 Relation to Congestion

In RM for hospitality industry such as RRM, degree of consumed resources is expressed by occupancy rate [16] which is an important management index. On the other hand, customers do not apparently have the pleasure of the occupancy rate. In the case, the occupancy rate can be regarded as congestion level from the customer's eye.

Facilities with seats in table have several conditions, such as a waiting line, reconstructable table and etc. A waiting line and reconstructable table are considered in [6] and [14], respectively.

However, congestion as a condition has not been focused on in the seating problem. In this section, from a viewpoint of congestion, we discuss the model suggested in this chapter.

A facility of which a person can see a state of the inside actually exists. Then, we can consider that a arriving rate of the person depends on a congestion level of the facility. The number of customer in the facility is included in the model of this chapter since states of the model preserve the number of each party. For each $X \in \bigcup_{n=1}^{N} X_n$, we define

$$f(X) = \sum_{p=1}^{\overline{P}} \sum_{i \in I_p} g_p x_p^i,$$

which stands for the number of people on a state $X$, and $f(X)$ is called the congestion level of $X$. Using this $f$, we obtain the following Proposition 2.11 which is a generalization of Lemma 2.4.

**Proposition 2.11.** *Let $p \in P$, $n \in T_0$ and $X \in X_n$. Suppose that Assumption 2.2 and 2.3 are satisfied, and that $\lambda(X') \geq \lambda(X'')$ for any $X', X'' \in X_n$ with $f(X') \leq f(X'')$. In addition, let $\delta,\ \delta' \in I_p$ with $\sum_p x_p^\delta < m_\delta$, and $\sum_p x_p^{\delta'} < m_{\delta'}$. If $t_\delta < t_{\delta'}$, then*

$$\Delta_p^\delta U_n(X) \leq \Delta_p^{\delta'} U_n(X). \tag{2.26}$$

***Proof:*** From (2.4),

$$\begin{aligned} U_n(X) = \sum_{p=1}^{\overline{P}} \lambda_p^n(f_X) \left\{ \left( r_p^n - \min_{i \in I_p} \Delta_p^i U_{n-1}(X) \right)^+ + U_{n-1}(X) \right\} \\ + \sum_{p=1}^{\overline{P}} \sum_{i \in I_p} x_p^i q_p^n U_{n-1}(X - \boldsymbol{e}_p^i) \\ + \left( 1 - \sum_{p=1}^{\overline{P}} \lambda_p^n(f_X) - \sum_{p=1}^{\overline{P}} \sum_{i \in I_p} x_p^i q_p^n \right) U_{n-1}(X), \\ X \in X_n, n \in T_1. \end{aligned} \tag{2.27}$$

Boundary conditions are that $U_n(X) = -\infty$ for $X \notin X_n, n \in T_0$, and $U_0(X) = 0$ for $X \in X_0$. Then, $\lambda_p^n(X + \boldsymbol{e}_p^\delta)$ in $U_n(X + \boldsymbol{e}_p^\delta)$ is equal to s $\lambda_p^n(X + \boldsymbol{e}_p^{\delta'})$ in $U_n(X + \boldsymbol{e}_p^{\delta'})$. Therefore, (2.26) can be proved similar to Lemma 2.4. ■

Allocating a party a table with larger size than the party size is called *upgrade*. Further, a structure that the maximum expected revenue does not increase when the manager of the facility upgrades is called the *upgrade structure*. Proposition 2.11 means that the upgrade structure is held if Assumption 2.1 is removed and the arrival rate is non-increasing function in the number of customers who are in the facility.

Consider a relationship between this upgrade structure and vacant seats. Upgrading an arriving party is increasing the number of seats of the party, that is, upgrading leads to increasing the party's space. We have already had optimal policies from (2.3). Here we provide variation of this optimal policies. For each $p \in P$, $n \in T_1$ and $X \in X_n$, we propose

$$\min \boldsymbol{I}^*_{np}(X) = \underline{i}^*_{np}(X), \tag{2.28}$$

$$\max \boldsymbol{I}^*_{np}(X) = \overline{i}^*_{np}(X) \tag{2.29}$$

where $\boldsymbol{I}^*_{np}(X) = \arg\min_{i \in I_p} \Delta^i_p U_{n-1}(X)$ An optimal policy which is produced by (2.28) is called a min-policy. Similarly, an optimal policy which is produced by (2.29) is called a max-policy. For these optimal policies, we can get the following Lemma 2.12 of the upgrade structure.

**Lemma 2.12.** *Let $n \in T_0$, $p \in P$, and $\delta,\ \delta' \in I_p$ with $\delta \neq \delta'$ and $t_\delta < t_{\delta'}$. In addition, let $X \in X_n$ with $\sum_p x^\delta_p < m_\delta,\ \sum_p x^{\delta'}_p < m_{\delta'}$. If a policy is the min-policy or max-policy, then*

$$\Delta^\delta_p U_n(X) \leq \Delta^{\delta'}_p U_n(X).$$

***Proof:*** Let $\underline{\boldsymbol{d}}^* = (\underline{d}^*_p)$ be the optimal policy vector which is given by (2.28). Similarly, let $\overline{\boldsymbol{d}}^* = (\overline{d}^*_p)$ be an optimal policy vector which is obtained by (2.29). We just show the case of applying $\overline{\boldsymbol{d}}^*$ since it is clearly for the case of applying $\underline{\boldsymbol{d}}^*$ from Lemma 2.4. In the case of applying $\overline{\boldsymbol{d}}^*$, we should see only the arrival parts because the departure and the null parts are the same as the ones of Lemma 2.4. Let $\underline{d}^{(\delta)*}_p$ and $\underline{d}^{(\delta')*}_p$ be $\underline{d}^*_p$ for $U_n(X + \boldsymbol{e}^\delta_p)$ and $U_n(X + \boldsymbol{e}^{\delta'}_p)$, respectively. In addition, let $\overline{d}^{(\delta)*}_p$ and $\overline{d}^{(\delta')*}_p$ be $\overline{d}^*_p$ for $U_n(X + \boldsymbol{e}^\delta_p)$ and $U_n(X + \boldsymbol{e}^{\delta'}_p)$, respectively.

i) $\overline{d}^{(\delta)*}_p \neq 0,\ \overline{d}^{(\delta')*}_p \neq 0$:

i - 1) $\overline{d}^{(\delta)*}_p \leq \overline{d}^{(\delta')*}_p$:

It is the same as the case $\underline{d}^{(\delta)*}_p \leq \underline{d}^{(\delta')*}_p$.

i - 2) $\overline{d}^{(\delta)*}_p > \overline{d}^{(\delta')*}_p$:

We divide the case of $\min I_p$ into five cases.

i - 2 - 1) $\delta' < \min I_p$:

From $\underline{d}_p^{(\delta)*} = \underline{d}_p^{(\delta')*}$, it follows that $r_p^n + U_{n-1}(X + \boldsymbol{e}_p^\delta + \boldsymbol{e}_p^{\overline{d}_p^{(\delta)*}}) = r_p^n + U_{n-1}(X + \boldsymbol{e}_p^\delta + \boldsymbol{e}_p^{\underline{d}_p^{(\delta)*}})$ $= r_p^n + U_{n-1}(X + \boldsymbol{e}_p^\delta + \boldsymbol{e}_p^{\underline{d}_p^{(\delta')*}}) = r_p^n + U_{n-1}(X + \boldsymbol{e}_p^\delta + \boldsymbol{e}_p^{\overline{d}_p^{(\delta')*}}) \geq r_p^n + U_{n-1}(X + \boldsymbol{e}_p^{\delta'} + \boldsymbol{e}_p^{\overline{d}_p^{(\delta')*}})$.

i - 2 - 2) $\delta' = \min I_p$:

Comparing $X + \boldsymbol{e}_p^\delta$ and $X + \boldsymbol{e}_p^{\delta'}$, we have that $\delta' = \underline{d}_p^{(\delta)*}$ and $\delta' \leq \overline{d}_p^{(\delta')*}$. Hence, $r_p^n + U_{n-1}(X + \boldsymbol{e}_p^\delta + \boldsymbol{e}_p^{\overline{d}_p^{(\delta)*}}) = r_p^n + U_{n-1}(X + \boldsymbol{e}_p^\delta + \boldsymbol{e}_p^{\underline{d}_p^{(\delta)*}}) = r_p^n + U_{n-1}(X + \boldsymbol{e}_p^\delta + \boldsymbol{e}_p^{\delta'}) \geq$ $r_p^n + U_{n-1}(X + \boldsymbol{e}_p^{\delta'} + \boldsymbol{e}_p^{\overline{d}_p^{(\delta')*}})$.

i - 2 - 3) $\delta < \min I_p < \delta'$:

It is the same as i - 2 - 1).

i - 2 - 4) $\delta = \min I_p$:

In this case, $\underline{d}_p^{(\delta)*} \leq \delta'$ and $\underline{d}_p^{(\delta')*} = \delta$. Hence, $r_p^n + U_{n-1}(X + \boldsymbol{e}_p^{\delta'} + \boldsymbol{e}_p^{\overline{d}_p^{(\delta')*}}) = r_p^n + U_{n-1}(X + \boldsymbol{e}_p^{\delta'} + \boldsymbol{e}_p^{\underline{d}_p^{(\delta')*}}) = r_p^n + U_{n-1}(X + \boldsymbol{e}_p^{\delta'} + \boldsymbol{e}_p^{\delta})$ $\leq r_p^n + U_{n-1}(X + \boldsymbol{e}_p^\delta + \boldsymbol{e}_p^{\underline{d}_p^{(\delta)*}}) = r_p^n + U_{n-1}(X + \boldsymbol{e}_p^\delta + \boldsymbol{e}_p^{\overline{d}_p^{(\delta)*}})$.

i - 2 - 5) $\delta > \min I_p$:

It is the same as i - 2 - 1).

For cases ii) $\overline{d}_p^{(\delta)*} = 0,\ \overline{d}_p^{(\delta')*} \neq 0$, iii) $\overline{d}_p^{(\delta)*} \neq 0,\ \overline{d}_p^{(\delta')*} = 0,$ and iv) $\overline{d}_p^{(\delta)*} = \overline{d}_p^{(\delta')*} = 0$, it is the same procedure as the case of $\underline{\boldsymbol{d}}_p$.

Then, applying $\overline{\boldsymbol{d}}^*$, it follows that $\Delta_p^\delta U_n(X) \leq \Delta_p^{\delta'} U_n(X)$. ■

From Lemma 2.12, Algorithm 1, which is similar to an algorithm suggested by Steinhardt and Gönsch [35], and Algorithm 2 are presented.

---

**Algorithm 1** An algorithm to calculate a min-policy.

---

Input $p, n$ and $X \in X_n$.
Solve $\underline{i}^* = \min\{i | i \in I_p, \sum_{p=1}^{\overline{P}} x_p^i < m_i\}$.
**if** $\underline{i}^*$ exists and $r_p^n \geq \Delta_p^{\underline{i}^*} U_{n-1}(X)$, **then**
  Accept the request of $p$ to $\underline{i}^*$.
**else**
  Deny the request of $p$.
**end if**

---

It is obvious that a computational cost of Algorithm 2 is larger that the one of Algorithm 1. Though we might consider that Algorithm 2 is useless, we can regard that Algorithm 2 upgrades as

**Algorithm 2** An algorithm to calculate a max-policy.

Input $p, n$ and $X \in X_n$.
$tmp \leftarrow 0$
Solve $\underline{i}^* = \min\{i | i \in I_p, \sum_{p=1}^{\overline{P}} x_p^i < m_i\}$.
**if** $\underline{i}^*$exits and $r_p^n \geq \Delta_p^{\underline{i}^*} U_{n-1}(X)$, **then**
  $tmp = \underline{i}^*$
  **while** $tmp + 1$ exits and $\Delta_p^{tmp} U_{n-1}(X) = \Delta_p^{tmp+1} U_{n-1}(X)$ **do**
    $tmp = tmp + 1$
  **end while**
  Accept the request of $p$ を to $tmp = \overline{i}^*$.
**else**
  Deny the request of $p$.
**end if**

much as possible while preserving the expected revenue.

Then, we propose two indices SP (Seats per a person) and SS (Surplus Seats). For each $p \in P$ and $X \in \bigcup_{n=0}^{N} X_n$, we define $SP_p(X)$ as

$$SP_p(X) = \begin{cases} \frac{\sum_{i \in I_p} x_p^i t_i}{\sum_{i \in I_p} x_p^i g_p} & \text{if} \sum_{i \in I_p} x_p^i \neq 0, \\ 1 & \text{if} \sum_{i \in I_p} x_p^i \neq 0, \end{cases}$$

which stands for average of the number of seats of the party, and $SS_p(X)$ as

$$SS_p(X) = \sum_{i \in I_p} x_p^i (t_i - q_p), \tag{2.30}$$

which stands for the number of surplus seats of the party. Remark that SP and SS are considered in an arriving point of a party.

For $p \in P$, $i \in I_p$ and $X \bigcup_{n=1}^{N} X_n$, $\Delta_p^i SP(X) = SP_p(X + \boldsymbol{e}_p^i) - SP_p(X)$ and $\Delta_p^i SS(X) = SS_p(X + \boldsymbol{e}_p^i) - SS_p(X)$ stand for deflections of SP and SS when a party $p$ is accepted to a table $i$ in a state $X$, respectively. Theorem 2.13 and Theorem 2.14, which relate to $\Delta_p^i SP(X)$ and $\Delta_p^i SS(X)$, respectively, are indicated following.

**Theorem 2.13.** *Under all assumptions in Lemma Lemma 2.12,* $\Delta_p^{\delta} SP(X) < \Delta_p^{\delta'} SP(X)$.

**Theorem 2.14.** *Under all assumptions in Lemma Lemma 2.12,* $\Delta_p^{\delta} SS(X) < \Delta_p^{\delta'} SS(X)$.

Proofs of these theorems are omitted because these are obvious from definitions of SS and SP.

Theorems 2.13 and 2.14 present trade-off relationships between SP or SS and the maximum expected revenue.

Robson and Kimes [31] reported that a party given an oversize table felt more comfortable than a right-size table. From this report, assume that it is true. Here, the maximum expected revenue on which a facility focuses and comfort on which customers focus have a trade-off relationship.

Using SP as an index, comfort decreases contrary to intuition when a party is accepted because the index SP takes an average. However, for SS, the problem dose not occur because the addition of the number of seats is only implemented. Therefore, we use only SS as an index for the number of seats which parties have.

From a viewpoint of customer's comfort, Algorithm 2 is better than Algorithm 1. In addition, consider a policy with $\Delta_p^i SS(X)$ for $n \in T_1, n \in X_n, p \in P$ and $i \in I_p$. Let $\alpha_p^n$ be coefficients which convert comfort which a party $p \in P$ obtains by an extra seat in $n \in T_1$ into a price. For each $n \in T_0$ and $X \in X_n$, we propose $\hat{U}_n(X)$ as maximum expected revenue which includes $\alpha_p^n, p \in P, n \in T_1$. $\hat{U}_n(X)$ are defined as follows.

$$\begin{aligned}\hat{U}_n(X) = \sum_{p=1}^{\overline{P}} \lambda_p^n \left\{ \left( \max_{i \in I_p} \left( r_p^n + \alpha_p^n \Delta_p^i SS(X) - \Delta_p^i \hat{U}_{n-1}(X) \right) \right)^+ + \hat{U}_{n-1}(X) \right\} \\ + \sum_{p=1}^{\overline{P}} \sum_{i \in I_p} x_p^i q_p^n \hat{U}_{n-1}(X - \boldsymbol{e}_p^i) \\ + \left( 1 - \sum_{p=1}^{\overline{P}} \lambda_p^n - \sum_{p=1}^{\overline{P}} \sum_{i \in I_p} x_p^i q_p^n \right) \hat{U}_{n-1}(X), \\ X \in X_n, n \in T_1.\end{aligned} \tag{2.31}$$

Boundary conditions are that $\hat{U}_n(X) = -\infty$ for $X \notin X_n, n \in T_0$, and $\hat{U}_0(X) = 0$ for $X \in X_0$.

Optimal policies of (2.31) are named SS-policy. The SS-policy is indicated as below.

**SS-policy :** An optimal policy for a party $p \in P$ and a state $X \in X_n$ in time $n \in T_1$ is to accept to allocate $\arg\max_{i \in I_p}(r_p^n + \alpha_p^n \Delta_p^i SS(X) - \Delta_p^i U_{n-1}(X))$ to the party $p$ if $\max_{i \in I_p}(r_p^n + \alpha_p^n \Delta_p^i SS(X) - \Delta_p^i U_{n-1}(X)) \geq 0$, or to deny to do so if $\max_{i \in I_p}(r_p^n + \alpha_p^n \Delta_p^i SS(X) - \Delta_p^i U_{n-1}(X)) < 0$.

Let $P_n(X)$ be a probability of being on a state $X \in X_n$ in $n \in T_0$, where $P_n(\mathbf{0}) = 1$,

$\mathbf{0} = (0, \cdots, 0| \cdots |0, \cdots, 0)$ and $P_n(X) = 0, n \notin T_0 \text{or} X \notin \bigcup_{n=0}^{N} X_n$. Here, a policy vector for a state $X \in X_n$ and time $n \in T_1$ presents $\boldsymbol{d}^n = (d_p^n) \in D_n(X)$. To simply, let $\boldsymbol{d}_{\delta\psi}^n = (d_{\delta\psi p}^n) \in D_n(X - \boldsymbol{e}_\psi^\delta)$ for $n \in T_1, X \in X_n, \psi \in P$ and $\delta \in I_\psi$. Then, we can denote $P_n(X)$ as a recessive form;

$$
\begin{aligned}
P_n(X) = P_{n+1}(X) \left\{ \sum_{p|d_p^{n+1}=0} \lambda_p^{n+1} + (1 - \sum_{p=1}^{\overline{P}} \lambda_p^{n+1} - \sum_{p=1}^{\overline{P}} \sum_{i \in I_p} x_p^i q_{ip}) \right\} \\
+ \sum_{p=1}^{\overline{P}} \sum_{i \in I_p} V_{ip}^n(X) \\
+ \sum_{p=1}^{\overline{P}} \sum_{i \in I_p} (x_p^i + 1) q_i p P_{n+1}(X + \boldsymbol{e}_p^i)
\end{aligned}
$$

for $n \in T_0 \backslash N, X \in X_n$, where $V_{ip}^n(X)$ indicates as

$$
V_{ip}^n(X) = \begin{cases} \lambda_p^{n+1} P_{n+1}(X - \boldsymbol{e}_p^i) & \text{if} X - \boldsymbol{e}_p^i \in X_{n+1}, d_{ipp}^{n+1} = i, \\ 0 & \text{otherwise.} \end{cases} \tag{2.32}
$$

Using this $P_n(X)$, the expectation of the number of excess seats is represented as below.

**Expectation of the number of excess seats in time** $n \in T_0$ **(ExSS)** :

$$
\sum_{X \in X_n} P_n(X) \sum_{p=1}^{\overline{P}} \sum_{i \in I_p} x_p^i (t_i - g_p).
$$

This ExSS is helpful to estimate degree of congestion which a policy generates in $n \in T_0$.

### 2.5.1 Numerical Examples

We observe revenue and congestion which $\alpha_p^n$ affect by computing ExSS from some policies; the min-policy, the max-policy and the SS-policy. Expected congestion level and ExSS can be obtained optimal policies throughout all $n \in T_0$ and $X \in X_n$. Then, let $\boldsymbol{d}_n(X) = (d_p) \in D_n(X), n \in T_1, X \in X_n$. In addition, Set $\boldsymbol{d}_n^*(X)$ as optimal ones for $n \in T_1$ and $X \in X_n$, and $\boldsymbol{d}_n^* = (\boldsymbol{d}_n^*(X))_{X \in X_n}, n \in T_1$. Let optimal policies for all time period, states, and arriving parties be $\Omega = (\boldsymbol{d}_N^*, \boldsymbol{d}_{N-1}^*, \cdots, \boldsymbol{d}_1^*)$. At this $\Omega$, let $\Omega_{min}$, $\Omega_{max}$ and $\Omega_{SS}$ be $\Omega$ calculated by the min-policy, the max-policy and the SS-policy, respectively. In addition, let $ExSS(\Omega)$ be ExSS which is obtained by a policy $\Omega$ from $n = N$ to $n = 0$.

Input data is as follows. $N = 100$. Configuration of parties and tables are $\overline{P} = 4,\ \overline{I} = 4,\ g_1 = 1, g_2 = 2, g_3 = 3, g_4 = 4,\ t_1 = 1, t_2 = 2, t_3 = 3, t_4 = 4\ , m_1 = 2, m_2 = 2\ , m_3 = 3$ and $m_4 = 4$. The arrival rate, departure rate and revenue for each time $n \in T_0$ are shown in Figure 2.3, Figure 2.4 and Figure 2.5, respectively. Further, $\alpha_1^n = 2, \alpha_2^n = 4, \alpha_3^n = 6, \alpha_4^n = 8, n \in T_0$.

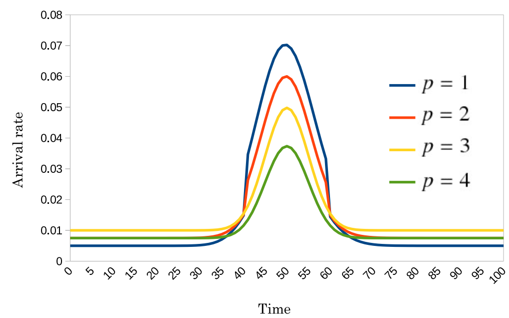


**Figure 2.3.** Arrival rate.

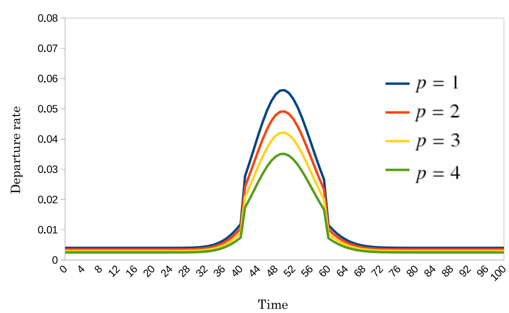


**Figure 2.4.** Departure rate.

Figure 2.6 presents ExSS for each time $n$ which is computed from the min-policy, the max-policy, and the SS-policy. The policies are obtained from the input data.

Getting closer to $n = 0$, $ExSS(\Omega_{SS})$ increases more than $ExSS(\Omega_{min})$ and $ExSS(\Omega_{max})$ from effect of $\alpha_p^n$. $ExSS(\Omega_{min})$ is similar to $ExSS(\Omega_{max})$ without at near closing time. The reason for this is what arriving requests do not occur after closing time. It is intuitive that difference between ExSS of the min-policy and the max-policy is generated at only near closing time even though the maximum revenues obtained by these policies are the same.

Then, consider revenue which includes SS. Let $X^0 = (0|0, 0|0, 0, 0|0, 0, 0, 0)$ be the initial state.

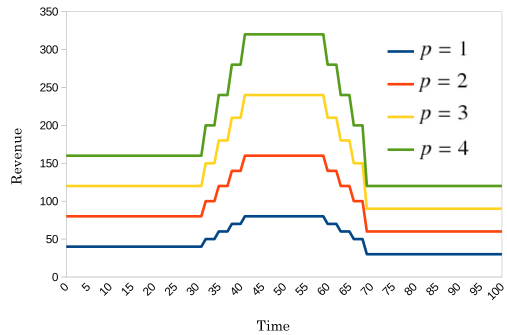


**Figure 2.5.** Revenue.

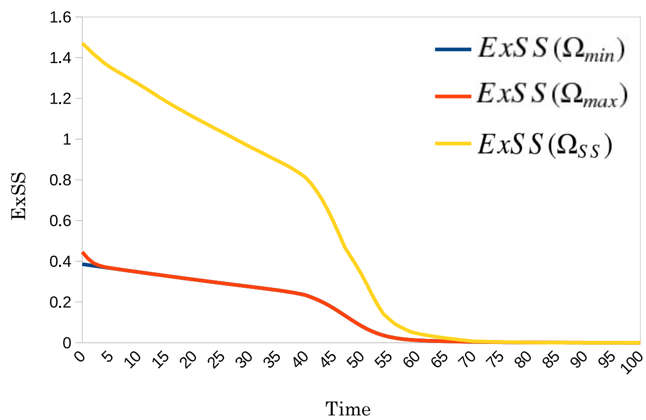


**Figure 2.6.** ExSS for max-policy and SS-policy.

For $X^0$, let $ExRev(\Omega_{min})$ and $ExRev(\Omega_{max})$ be the included revenue from $n = N$ to $n = 0$ using policies $\Omega_{min}$ and $\Omega_{max}$, respectively. In the following Figure 2.7, $\hat{U}_n(X^0)$ and $ExRev(\Omega_{min})$ for each $n \in T_0$ are shown, where $ExRev(\Omega_{max})$ is not presented because it is almost the same as $ExRev(\Omega_{min})$.

Seeing Figure 2.7, we can find that $ExRev(\Omega_{min})$ decreases more than $\hat{U}_n(X^0)$ on $n \in T_0$. It means that the min-policy sacrifices a lot of customers' comfort if The number of customers' seats and customers' comfort have positive correlation.

## 2.6 Relation to Other Models and Current RM's Problems

$\alpha_p^n, p \in P, n \in T_0$ introduced in the previous section are apparently seen as additional revenue that the facility can obtain later, or values of estimate for customers' comfort that the facility want to give for the customers. However, it is actually difficult to estimate $\alpha_p^n, p \in P, n \in T_0$.

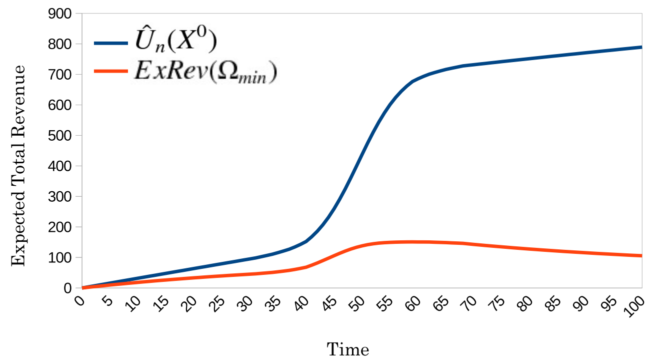


**Figure 2.7.** Expected total revenue.

Arriving parties are segmented by only their sizes in this model. It may be better to consider relationships a facility and the customers, additionally. The examples of the customer with the relationships are walk-in customer or regular customer. It means that the model imports Customer Relationship Management (CRM). This necessity of integration between RM and CRM is reported as RM's challenge in [41].

Then, we state about the model from a viewpoint of the additional revenue, in other words, long-term revenue. In RM, decision-making with short-term revenue and long-term revenue have been dealt with by Klein and Kolb [24]. They proposed a model with arriving customers who change their behavior by a facility's decision. However, we know that it is difficult to apply this approach to the dynamic model in this chapter because of enlargement of computational cost.

## 2.7 Conclution

In this chapter, we introduced a dynamic model with seats in tables and its structural properties. Further, we mentioned to take congestion level or surplus seat in the model.

Theorem 2.9 indicates that differences of departure rate among parties cause to increase the variations of optimal policies, that is, increase requisite data capacity for optimal policies of each party and time period. If the parameters set is that $\overline{P} = 4,\ \overline{I} = 2,\ g_p = p,\ t_1 = 2, t_2 = 4,\ m_1 = 6,$ and $m_2 = 7$ where $p \in P$, then $\chi = 9240$ in (2.1). However, if there is not difference in departure rates for the case, the maximum of the variations of the optimal policies is reduced to $56$.

As for adopting congestion level and reward for upgrade, it leads to have variation of optimal policies; min-policy, max-policy and SS-policy. Applying the min-policy or max-policy, we can use upgrade structure in this model. Further, it is held in the case where more congested makes lower arriving rates of parties.

# List of Symbols for Chapter 3

$P$: set of size of arriving parties

$\overline{P}$: the largest size in $P$

$\overline{C}$: the number of resources

$C$: set of the resources; $C = \{1, \cdots, \overline{C}\}$

$N$: size of time period

$T_0$: set of time periods $\{0, 1, \cdots, N\}$

$T_1$: set of time periods $\{1, \cdots, N\}$

$r_p^n$: fare of a request $p \in P$ in $n \in T_1$

$\lambda_p^n(X)$: arrival rate of the party $p \in P$ in $n \in T_1$

$\lambda_0^n$: rate of no-event in $n \in T_1$

$X_n$: state space in $n \in T_0$

$(a, b) \in \boldsymbol{A}_p(X)$: policy space for the request $p \in P, n \in T_1$ and $X \in X_n$ where $a$ and $b$ are a size of segment and an index of the segment $a$, respectively

$U_n(X)$: maximum expected revenue from operating over periods $n$ to 0 in the state $X \in X_n$

$G$: set of fare classes

$J$: set of kinds of arriving requests; $J = P \times G$

$\hat{U}_n(X)$: maximum expected revenue with multi-fare classes

$d \in \hat{\boldsymbol{A}}_p(X)$: policy space for the request $p \in P, n \in T_1$ and $X \in X_n$ where $d$ is a size of segment

$\Delta_p^d U_n(X)$: opportunity cost of accepting the party $p \in P$ for the segment $d$;
$\Delta_p^d U_n(X) = U_n(X) - U_n(X + \boldsymbol{e}_{d-p} - \boldsymbol{e}_d)$

# Chapter 3

# Single Line Seats Model

## 3.1 Introduction

When we book a seat on a plane or bullet train, we can often choose seat location. However, the assignment of seat location is more important if a facility is a theater or a stadium. These facilities are shown as applications of revenue management in Kimes and Wirtz [23].

In this chapter, consider a dynamic model with seats placed on lines, such as a sushi bar (bar counter), theater, or stadium. Then, the model is taken account of seating position. However, if we identify all positions, then the number of elements of the state space is $2^H$, where $H$ means the number of seats, and the computation is actually impossible. Hence, we reduce the state space and decision space by utilizing an approach of formulation for a pyramid game in [18]. Then, we consider a simple case in which the number of line is one. A model which deals with this case is called the *single line seats model*.

## 3.2 Formulation for Single Line Seats Model

Batch request $p \in P = \{1, \cdots, \overline{P}\}$ arrives for booking resources which are placed on a single line during the booking horizon, where request $p$ has a request demanding $p$ resources. The resources can be regarded as seats at a counter in a restaurant, a part of seats on a plane or bullet train or a part of seats in a theater or stadium. Let $\overline{C}$ be the number of the resources. Set $C = \{1, \cdots, \overline{C}\}$. Suppose that the batch requests cannot be separated and need adjacent available resources which are equal in size when the resources are allocated to the request. Let the booking horizon be sufficiently discreted into $N$ time periods so that no more than one request arrives in each period $n \in \{1, \cdots, N\}$. The time period progresses from $N$ to 1, and 0 is terminal time on the booking horizon. Suppose that there is a single fare class. (A case for multi fare class is mentioned in the next section.) Let $T_0$ and $T_1$ be $\{0, \cdots, N\}$ and $\{1, \cdots, N\}$, respectively. The fare depends on the size of the request and the time period. Let $r_p^n$ be a fare of the request $p \in P$ in time period $n \in T_1$. Arrival rates of the requests also depend on the size of requests and the time period. Let $\lambda_p^n$ be an arrival rate of the request $p \in P$ in time period $n \in T_1$. We set $\lambda_0^n = 1 - \sum_{p \in P} \lambda_p^n$, and $\lambda_0^n$ stands for rate of no-event in $n \in T_1$. Cancellation, overbooking and walk-in are ignored. To simplify the state of the resources, we do not distinguish between left and right side of the resources' state. Further, if we identify every position on the resources, then it is irrelevant because of generating various states to which the same requests can be accepted. However, this redundancy of the states can be removed, as shown next. Let $a_1, a_2, \cdots, a_{\overline{C}}$ be states of resources with size $\overline{C}$ where $a_k = 1$ if the resource is booked and $a_k = 0$ if the resource is unbooked, $a_0 = 1$ and $a_{\overline{C}+1} = 1$. When there are $a_k, a_{k+1}, \cdots, a_l (1 \le k \le l \le \overline{C})$ and $a_{k-1} = 1, a_k = a_{k+1} = \cdots = a_l = 0, a_{l+1} = 1$, we call $a_k, a_{k+1}, \cdots, a_l$ a segment of size $l - k + 1$. Let $X_n$ be the state space at $n \in T_0$ which is defined as

$$X_n = \left\{ (t_1, \cdots, t_{\overline{C}}) | 0 \le t_a \le \lfloor \frac{\overline{C}+1}{a+1} \rfloor, t_a \in \mathbb{Z}_+, a \in C, \sum_{a=1}^{\overline{C}} a t_a \le \overline{C} - \overline{P}(N-n), \right\}$$

for each $n \in T_0$, where $t_a$ is the number of segments of size $a$.

Note that the state space is reduced by $\overline{P}(N - n)$ because no more than one request arrives in the each period. Figure 3.1 shows variations of booked positions of the state $X = (0, 0, 1, 1, 0, 0, 0, 0, 0, 0)$. A shaded cell in Figure 3.1 indicates a booked resource.

Let $\boldsymbol{A}_p(X)$ be the policy space for a request $p \in P$, $n \in T_1$ and $X \in X_n$ which is defined as

$$\boldsymbol{A}_p(X) = \left\{ (a, b) | (b = 0) \text{or} (t_a > 0, p \ge a, 0 < b \le \frac{a - p + 2}{2}, b \in \mathbb{Z}_+, a \in C) \right\}. \tag{3.1}$$

The condition $b > 0$ in the policy space indicates an index of a position in a segment $a$ and $b = 0$ means to deny a request $p$. The condition $0 < b \le \frac{a-p+2}{2}$ in the policy space is briefly explained

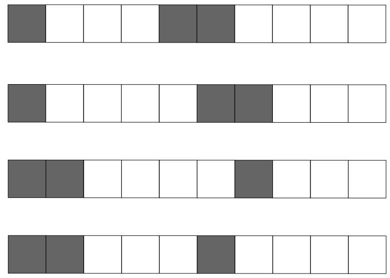

**Figure 3.1.** Variations of the state $X = (0, 0, 1, 1, 0, 0, 0, 0, 0, 0)$.

as follows. We can easily recognize $0 < b < \frac{a}{2} + 1$ because the right side and the left side of the resources are indistinguishable. If $A_p = (a, b), b > 0$, then the range of index of the booked resources is from $b$ to $b + p - 1$ and a relational expression $b + p - 1 \leq a - b + 1$ is established where $a - b + 1$ is an inverted edge of $b$.

Figure 3.2 shows policies on which a request $p = 1$ is accepted to $a = 4$ for a state $X = (0, 0, 1, 1, 0, 0, 0, 0, 0, 0)$. As shown in Figure 3.2, we can confirm that the state can be modified if the policy is changed.

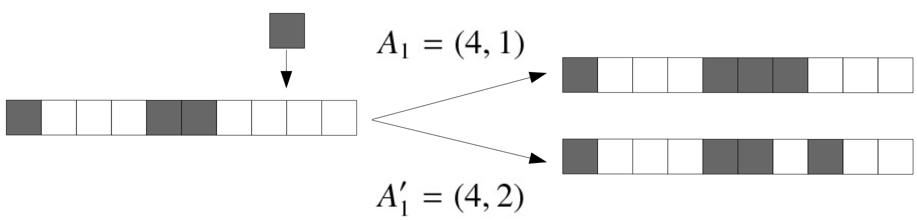


**Figure 3.2.** $A_1 = (4, 1)$ and $A_1' = (4, 2)$ for the state $X = (0, 0, 1, 1, 0, 0, 0, 0, 0, 0)$.

Let $U_n(X)$ be maximum expected revenue which a facility with the resources on initial state $X \in X_n$ can be obtained from optimally operating over the period $n$ to 0. $U_n(X)$ are shown as following equation:

$$U_n(X) = \sum_{p \in P} \lambda_p^n \max_{(a,b) \in \boldsymbol{A}_p(X)} \Big\{ \sum_{b|b \neq 0} (r_p^n + U_{n-1}(X + \boldsymbol{e}_{b-1} + \boldsymbol{e}_{a-(p+b-1)} - \boldsymbol{e}_a)) + \sum_{b|b=0} U_{n-1}(X) \Big\}$$
$$+ \lambda_0^n U_{n-1}(X),$$
$$X \in X_n, n \in T_1 \qquad (3.2)$$

where $\boldsymbol{e}_i = (e_1^i, \cdots, e_{\overline{C}}^i), i \in 0 \cup C$, $e_x^i = 1$ if $x = i$ and $1 \leq i \leq \overline{C}$, $e_x^i = 0$ if $x \neq i$ and $1 \leq i \leq \overline{C}$, and $e_1^0 = e_2^0 = \cdots = e_{\overline{C}}^0 = 0$. Boundary conditions for (3.2) are $U_n(X) = -\infty$, $X \notin X_n$ and $U_0(X) = 0$.

## 3.3 Properties of Single Line Seats Model

It is difficult to find optimal policies and the maximum expected revenue $U_n(X)$ because a size of the policy space enlarges exponentially. However, we can reduce the search range of the policy space by using the following Proposition 3.1.

**Proposition 3.1.** *Given a request $p \in P$ and $X \in X_n$ for $n \in T_1$, if there exists a policy $A_p = (a, b) \in \boldsymbol{A}_p(X)$ such that $b \geq 1$, then*

$$\begin{aligned} \max_{(a,b)\in \boldsymbol{A}_p(X)} \sum_{b|b\neq 0} & \left(r_p^n + U_{n-1}(X + \boldsymbol{e}_{b-1} + \boldsymbol{e}_{a-(p+b-1)} - \boldsymbol{e}_a)\right) \\ &= \max_{(a,1)\in \boldsymbol{A}_p(X)} \left(r_p^n + U_{n-1}(X + \boldsymbol{e}_{a-p} - \boldsymbol{e}_a)\right). \end{aligned} \tag{3.3}$$

***Proof:*** Given a request $p' \in P$ and $X \in X_n$ for $n \in T_0 \backslash N$, for Proposition 3.1, it should be shown that if there exists a policy $(a, b) \in \boldsymbol{A}_{p'}(X)$ such that $b \geq 1$, then

$$U_n(X + \boldsymbol{e}_{\psi-1} + \boldsymbol{e}_{\delta-(p'+\psi-1)} - \boldsymbol{e}_\delta) \leq U_n(X + \boldsymbol{e}_{\delta-p'} - \boldsymbol{e}_\delta)$$

for any $(\psi, \delta) \in \boldsymbol{A}_{p'}(X)$ by the inductive method. For the case $n = 0$, it is obvious that $U_0(X + \boldsymbol{e}_{\psi-1} + \boldsymbol{e}_{\delta-(p'+\psi-1)} - \boldsymbol{e}_\delta) = U_0(X + \boldsymbol{e}_{\delta-p'} - \boldsymbol{e}_\delta) = 0$ from the boundary conditions. Then, assume that $U_{n-1}(X + \boldsymbol{e}_{\psi-1} + \boldsymbol{e}_{\delta-(p'+\psi-1)} - \boldsymbol{e}_\delta) \leq U_{n-1}(X + \boldsymbol{e}_{\delta-p'} - \boldsymbol{e}_\delta), (\psi, \delta) \in \boldsymbol{A}_{p'}(X)$. Fix any $(\psi, \delta) \in \boldsymbol{A}_{p'}(X)$. To simplify notation, set $X^{(\psi)} = X + \boldsymbol{e}_{\psi-1} + \boldsymbol{e}_{\delta-(p'+\psi-1)} - \boldsymbol{e}_\delta$ and $X^{(1)} = X + \boldsymbol{e}_{\delta-p'} - \boldsymbol{e}_\delta$. It can be shown easily that

$$\begin{aligned} U_n(X^{(\psi)}) = \quad & \sum_{p\in P} \lambda_p^n \max_{(a^{(\psi)}, b^{(\psi)})\in \boldsymbol{A}_p(X^{(\psi)})} \left\{ \sum_{b^{(\psi)}|b^{(\psi)}\neq 0} \left(r_p^n + U_{n-1}(X^{(\psi)} + \boldsymbol{e}_{a^{(\psi)}-p} - \boldsymbol{e}_{a^{(\psi)}})\right) \right. \\ & \left. + \sum_{b^{(\psi)}|b^{(\psi)}=0} U_{n-1}(X^{(\psi)}) \right\} + \lambda_0^n U_{n-1}(X^{(\psi)}) \end{aligned}$$

and

$$U_n(X^{(1)}) = \sum_{p\in P} \lambda_p^n \max_{(a^{(1)},b^{(1)})\in \boldsymbol{A}_p(X^{(1)})} \left\{ \sum_{b^{(1)}|b^{(1)}\neq 0} (r_p^n + U_{n-1}(X^{(1)} + \boldsymbol{e}_{a^{(1)}-p} - \boldsymbol{e}_{a^{(1)}})) + \sum_{b^{(1)}|b^{(1)}=0} U_{n-1}(X^{(1)}) \right\} + \lambda_0^n U_{n-1}(X^{(1)}).$$

Let $A_p^*(X^{(\psi)}) = (a^{(\psi)*}, b^{(\psi)*})$ be an optimal policy for a request $p \in P$ and $X^{(\psi)}$. Likewise, let $A_p^*(X^{(1)}) = (a^{(1)*}, b^{(1)*})$ be an optimal policy for a request $p \in P$ and $X^{(1)}$.

i) $b^{(\psi)*} \neq 0, b^{(1)*} \neq 0$:

We should make a comparison between $U_{n-1}(X+\boldsymbol{e}_{\psi-1}+\boldsymbol{e}_{\delta-(p'+\psi-1)}-\boldsymbol{e}_\delta+\boldsymbol{e}_{a^{(\psi)*}-p}-\boldsymbol{e}_{a^{(\psi)*}})$ and $U_{n-1}(X+\boldsymbol{e}_{\delta-p'}-\boldsymbol{e}_\delta+\boldsymbol{e}_{a^{(1)*}-p}-\boldsymbol{e}_{a^{(1)*}})$. Furthermore, this case is divided in two cases because the case $X^{(\psi)} + \boldsymbol{e}_{a^{(\psi)*}-p} - \boldsymbol{e}_{a^{(\psi)*}} \notin X_{n-1}$ exists if $\psi - 1$ and $\delta - (p' + \psi - 1)$ are changed in the case: $a^{(\psi)*} = \psi - 1 \vee a^{(\psi)*} = \delta - (p' + \psi - 1)$.

i-1) $a^{(\psi)*} \neq \psi - 1, a^{(\psi)*} \neq \delta - (p' + \psi - 1)$:

From the condition of this case and the inductive hypothesis, $U_{n-1}(X + \boldsymbol{e}_{\psi-1} + \boldsymbol{e}_{\delta-(p'+\psi-1)} -\boldsymbol{e}_\delta+\boldsymbol{e}_{a^{(\psi)*}-p}-\boldsymbol{e}_{a^{(\psi)*}}) \leq U_{n-1}(X+\boldsymbol{e}_{\delta-p'}-\boldsymbol{e}_\delta+\boldsymbol{e}_{a^{(\psi)*}-p}-\boldsymbol{e}_{a^{(\psi)*}}) \leq U_{n-1}(X+\boldsymbol{e}_{\delta-p'}-\boldsymbol{e}_\delta+\boldsymbol{e}_{a^{(1)}-p} -\boldsymbol{e}_{a^{(1)}})$.

i-2) $a^{(\psi)*} = \psi - 1$ or $a^{(\psi)*} = \delta - (p' + \psi - 1)$:

$U_{n-1}(X + \boldsymbol{e}_{\psi-1} + \boldsymbol{e}_{\delta-(p'+\psi-1)} - \boldsymbol{e}_\delta + \boldsymbol{e}_{a^{(\psi)*}-p} - \boldsymbol{e}_{a^{(\psi)*}})\Big|_{a^{(\psi)*}=\psi-1 \text{ or } a^{(\psi)*}=\delta-(p'+\psi-1)}$ $\leq U_{n-1}(X+\boldsymbol{e}_{\delta-p'-g_p}-\boldsymbol{e}_\delta) = U_{n-1}(X+\boldsymbol{e}_{\delta-p'}-\boldsymbol{e}_\delta+\boldsymbol{e}_{\delta-p'-g_p}-\boldsymbol{e}_{\delta-p'}) \leq U_{n-1}(X+\boldsymbol{e}_{\delta-p'}-\boldsymbol{e}_\delta+ \boldsymbol{e}_{a^{(1)*}-g_p} - \boldsymbol{e}_{a^{(1)*}})$.

Therefore, $U_n(X^{(\psi)}) \leq U_n(X^{(1)})$ is obtained in the case i) from i-1) and i-2).

ii) $b^{(\psi)*} = 0, b^{(1)*} \neq 0$:

We should make a comparison between $U_{n-1}(X + \boldsymbol{e}_{\psi-1} + \boldsymbol{e}_{\delta-(p'+\psi-1)} - \boldsymbol{e}_\delta)$ and $r_p^n + U_{n-1}(X + \boldsymbol{e}_{\delta-p'} - \boldsymbol{e}_\delta + \boldsymbol{e}_{a^{(1)*}-p} - \boldsymbol{e}_{a^{(1)*}})$. From the fact $b^{(1)*} \neq 0$, $r_p^n+U_{n-1}(X+\boldsymbol{e}_{\delta-p'}-\boldsymbol{e}_\delta+\boldsymbol{e}_{a^{(1)*}-p}-\boldsymbol{e}_{a^{(1)*}}) \geq U_{n-1}(X+\boldsymbol{e}_{\delta-p'}-\boldsymbol{e}_\delta) \geq U_{n-1}(X+\boldsymbol{e}_{\psi-1}+\boldsymbol{e}_{\delta-(p'+\psi-1)} - \boldsymbol{e}_\delta)$. Therefore, $U_n(X^{(\psi)}) \leq U_n(X^{(1)})$ is obtained in this case ii).

iii) $b^{(\psi)*} \neq 0, b^{(1)*} = 0$:

We should compare $r_p^n + U_{n-1}(X + \boldsymbol{e}_{\psi-1} + \boldsymbol{e}_{\delta-(p'+\psi-1)} - \boldsymbol{e}_\delta + \boldsymbol{e}_{a^{(\psi)*}-p} - \boldsymbol{e}_{a^{(\psi)*}})$ with $U_{n-1}(X + \boldsymbol{e}_{\delta-p'} - \boldsymbol{e}_\delta)$. This case is divided in two cases from a reason which is the same as the one in the case i).

iii-1) $a^{(\psi)} \neq \psi - 1$, $a^{(\psi)} \neq \delta - (p' + \psi - 1)$:

$$r_p^n + U_{n-1}(X + \boldsymbol{e}_{\psi-1} + \boldsymbol{e}_{\delta-(p'+\psi-1)} - \boldsymbol{e}_\delta + \boldsymbol{e}_{a^{(\psi)*}-p} - \boldsymbol{e}_{a^{(\psi)*}}) \leq r_p^n + U_{n-1}(X + \boldsymbol{e}_{\delta-p'} - \boldsymbol{e}_\delta + \boldsymbol{e}_{a^{(\psi)*}-p}$$

$- \boldsymbol{e}_{a^{(\psi)*}}) \leq U_{n-1}(X + \boldsymbol{e}_{\delta-p'} - \boldsymbol{e}_\delta)$ is obtained.

iii-2) $a^{(\psi)} = \psi - 1 \text{or} a^{(\psi)} = \delta - (p' + \psi - 1)$:

$$r_p^n + U_{n-1}(X + \boldsymbol{e}_{\psi-1} + \boldsymbol{e}_{\delta-(p'+\psi-1)} - \boldsymbol{e}_\delta + \boldsymbol{e}_{a^{(\psi)*}-p} - \boldsymbol{e}_{a^{(\psi)*}})\Big|_{a^{(\psi)*}=\psi-1 \text{ or } a^{(\psi)*}=\delta-(p'+\psi-1)}$$

$$\leq r_p^n + U_{n-1}(X + \boldsymbol{e}_{\delta-p'-g_p} - \boldsymbol{e}_\delta) = U_{n-1}(X + \boldsymbol{e}_{\delta-p'} - \boldsymbol{e}_\delta + \boldsymbol{e}_{\delta-p'-g_p} - \boldsymbol{e}_{\delta-p'}) \leq U_{n-1}(X + \boldsymbol{e}_{\delta-p'} - \boldsymbol{e}_\delta).$$

Therefore, $U_n(X^{(\psi)}) \leq U_n(X^{(1)})$ is obtained in the case iii) from the cases iii-1) and iii-2).

iv) $b^{(\psi)*} = 0, b^{(1)*} = 0$:

$U_n(X^{(\psi)}) \leq U_n(X^{(1)})$ is easily obtained from the inductive hypothesis.

Consequently, Proposition 3.1 is proofed from the cases: i) - vi). ■

Note that Proposition 3.1 does not related to fares $r_p^n$. This fact shows that Proposition 3.1 is achieved in any orderings of the fares $r_p^n$ among requests $p \in P$. Further, this is important for multi-fare classes case which is mentioned below.

We additionally show maximum expected revenue for a case with multi-fare classes. Let $G = \{1, \cdots, \overline{G}\}$ be a set of fare class. Set $J = P \times G$ as kinds of arriving request. In this multi-fare class case, the state space does not need to be changed from the one of the case of single fare class. At its decision space, its definition does not be changed because only an arriving party's size relates to consumption for seats. Thus, maximum expected revenue with multi-fare class is the following equation.

$$\hat{U}_n(X) = \sum_{(p,g)\in J} \lambda_{(p,g)}^n \max_{(a,b)\in \boldsymbol{A}_p(X)} \left\{ \sum_{b|b\neq 0} (r_{(p,g)}^n + \hat{U}_{n-1}(X + \boldsymbol{e}_{b-1} + \boldsymbol{e}_{a-(p-b-1)} - \boldsymbol{e}_a)) \right.$$
$$\left. + \sum_{b|b=0} \hat{U}_{n-1}(X) \right\} + \lambda_0^n \hat{U}_{n-1}(X)$$
$$, X \in X_n, n \in T_1. \quad (3.4)$$

Boundary conditions for (3.4) are $\hat{U}_n(X) = -\infty$, $X \notin X_n$ and $\hat{U}_0(X) = 0$ A similar property to Proposition 3.1 is obtained in (3.4).

**Proposition 3.2.** *Given a request $(p, g) \in J$ and $X \in X_n$ for $n \in T_1$, if there exists a policy*

$A_p = (a, b) \in \boldsymbol{A}_p(X)$ *such that* $b \geq 1$, *then*

$$\max_{(a,b)\in \boldsymbol{A}_p(X)} \sum_{b|b\neq 0} \left(r^n_{(p,g)} + \hat{U}_{n-1}(X + \boldsymbol{e}_{b-1} + \boldsymbol{e}_{a-(p+b-1)} - \boldsymbol{e}_a)\right)$$
$$= \max_{(a,1)\in \boldsymbol{A}_p(X)} \left(r^n_{(p,g)} + \hat{U}_{n-1}(X + \boldsymbol{e}_{a-p} - \boldsymbol{e}_a)\right). \tag{3.5}$$

***Proof:*** Procedure for this proof is the same as Proposition 3.1's one. ■

Proposition 3.1 and Proposition 3.2 also indicate that we should deal with only $(a, b) \in \boldsymbol{A}_p(X), b = 1$ or $b = 0$ for a request $p$ (or $(p, g)$) and state $X \in X_n$ to solve the maximum expected revenue. This leads to a remarkable fact for the case $X_N = \{\boldsymbol{e}_{\overline{C}}\}$ which is that all resources are not occupied with requests at the beginning of the booking horizon. Details of this fact are explained in the following Remark 3.3.

**Remark 3.3.** Let $\hat{X} \in \hat{X}_n, n \in T_0$ be the set of states which are obtained when the maximum expected revenue is solved in the case $X_N = \{\boldsymbol{e}_{\overline{C}}\}$. From Proposition 3.2 and the condition of $X_N$, $\hat{X}_n = \{\boldsymbol{e}_x | 0 \leq x \leq \overline{C} - \overline{P}(N - n)\}, n \in T_0$. The $x$ can be regarded as the number of unbooked resources. Set $\hat{V}_n(x) = \hat{U}_n(\boldsymbol{e}_x)$ and $\boldsymbol{A'}_p(x) = \{b | (x \geq p, b = 1)$ or $b = 0\}, x \in 0 \cup C$. Then, it follows that

$$\hat{V}_n(x) = \sum_{(p,g)\in J} \lambda^n_{(p,g)} \max\left\{r^n_{(p,g)} + \hat{V}_{n-1}(x - p), \hat{V}_{n-1}(x)\right\} + \lambda^n_0 \hat{V}_{n-1}(x),$$
$$0 \leq x \leq \overline{C}, n \in T_1. \tag{3.6}$$

Boundary conditions are $\hat{V}_0(x) = 0$ and $\hat{V}_n(x) = -\infty, x < 0$.

(3.6) can be regarded as a single-leg model with multiple booking class and multiple seat booking in [26]. This fact indicates that the maximum expected revenue and the optimal policy which is obtained by the single-leg model is the same as the ones which is obtained by the model of this paper for the case $X_N = \{\boldsymbol{e}_{\overline{C}}\}$. However, (3.6) cannot be applied to the case: $X_N \neq \{\boldsymbol{e}_{\overline{C}}\}$.

We deal with only single fare class case as below to simplify. From Proposition 3.1, the policy space can be rewritten in the following form because $a$ in $\boldsymbol{A}_p(X)$ is only decided if a booking request $p$ is accepted.

$$\hat{\boldsymbol{A}}_p(X) = \{d | (d = a, t_a \geq 0, a \geq p) \text{ or } (d = 0)\}.$$

$d = 0$ in the policy space stands for denying a request. By using this $\hat{\boldsymbol{A}}_p(X)$, (3.2) can be rewritten as the following equation:

$$\begin{aligned} U_n(X) = \quad & \sum_{p \in P} \lambda_p^n \max_{d \in \hat{\boldsymbol{A}}_p(X)} \left\{ \sum_{d|d \neq 0} (r_p^n + U_{n-1}(X + \boldsymbol{e}_{d-p} - \boldsymbol{e}_d)) + \sum_{d|d=0} U_{n-1}(X) \right\} \\ & + \lambda_0^n U_{n-1}(X), \\ & X \in X_n, n \in T_1. \end{aligned} \tag{3.7}$$

Further, (3.7) is rewritten in the following equation

$$\begin{aligned} U_n(X) = \sum_{p \in P} \lambda_p^n \left( r_p^n - \min_{d \in \hat{\boldsymbol{A}}_p(X)} \Delta_p^d U_{n-1}(X) \right)^+ + U_{n-1}(X), \\ X \in X_n, n \in T_1. \end{aligned} \tag{3.8}$$

where $(k)^+ = \max\{k, 0\}$, $\Delta_p^d U_{n-1}(X) = U_{n-1}(X) - U_{n-1}(X + \boldsymbol{e}_{d-p} - \boldsymbol{e}_d)$ and $\Delta_p^0 U_n(X) = \infty$. $\min_{d \in \hat{\boldsymbol{A}}_p(X)} \Delta_p^d U_{n-1}(X)$ can be seen as threshold price. From (3.8), it is obvious to acquire the following Theorem 3.4.

**Theorem 3.4.** *For a given $X \in X_0$, $U_n(X)$ is non-decreasing in $n$.*

Then, an algorithm is shown as below. Algorithm 3 is to calculate an optimal policy $d^*$ for a given $p$, $n$ and $X \in X_n$ by using the threshold price. $\hat{\boldsymbol{A}}_p(X) \backslash 0$ in Algorithm 3 stands for $\hat{\boldsymbol{A}}_p(X) \backslash \{0\}$.

If $\arg\min_{d \in \hat{\boldsymbol{A}}_p(X) \backslash 0} \Delta_p^d U_{n-1}(X)$ is not unique, then without loss of generality, the smallest $d$ is selected in Algorithm 3. Note that we cannot reduce the search range for segments $a$ of $\boldsymbol{A}_p(X)$ in Algorithm 3.

We might expect that there is a structural property of the upgrade model which is seen in Steinhardt and Gönsch [35] and Lemma 2.4. This means that, for all $n \in T_0$ and $X \in X_n$, if there exist $d \in \hat{\boldsymbol{A}}_p(X)$ and $d' \in \hat{\boldsymbol{A}}_p(X)$ such that $d \neq 0, d' \neq 0$ and $d \leq d'$, then $\Delta_p^d U_n(X) \leq \Delta_p^{d'} U_n(X)$. Indeed, the monotonicity does not exist. To demonstrate the non-monotonicity, a counter-example is shown in the next section.

**Algorithm 3**

```
Input n, X ∈ X_n and p.
d* ← 0
if Â_p(X)\0 ≠ φ then
   calculate min_{d∈Â_p(X)\0} Δ_p^d U_{n−1}(X) and
   min_d{arg min_{d∈Â_p(X)\0} Δ_p^d U_{n−1}(X)}.
   d̄* ← min_d{arg min_{d∈Â_p(X)\0} Δ_p^d U_{n−1}(X)}
   if r_p^n ≥ min_{d∈Â_p(X)\0} Δ_p^d U_{n−1}(X) then
      d* ← d̄*
   end if
end if
```

## 3.4 Numerical Examples

This section shows numerical examples in which maximum expected revenues and optimal policies are calculated by using Algorithm 3 on a small scale. One of the examples is the counter-example which was mentioned in the previous section.

### 3.4.1 Numerical Example Using Algorithm 3

A data-set is $\overline{C} = 6, N = 4, \overline{P} = 3, X_4 = \{(0,1,1,0,0,0)\}$. Arrival rates and fares are shown in Table 3.1. From $\max_n |X_n| = 9$, to simplify notation, elements of the state space are $X^1 = (0,1,1,0,0,0), X^2 = (1,0,1,0,0,0), X^3 = (0,0,1,0,0,0), X^4 = (0,2,0,0,0,0), X^5 = (1,1,0,0,0,0), X^6 = (2,0,0,0,0,0), X^7 = (0,1,0,0,0,0), X^8 = (1,0,0,0,0,0), X^9 = (0,0,0,0,0,0)$.

The maximum expected revenues and the optimal policies which are calculated from the data-set are shown in Table 3.2 and 3.3, respectively. Specifically, a calculation process for $p = 1$ and $X^2 \in X_3$ in the Table 3.3 is explained as follows.

Suppose that calculations are terminated at $n = 2$ by backward induction. We calculate for $d = 1$ because of $\hat{\boldsymbol{A}}_1(X^2)\backslash 0 \neq \phi$. For a policy $1 \in \hat{\boldsymbol{A}}_1(X^2)$, $\Delta_1^1 U_2(X^2) = U_2(X^2) - U_2(X^3) = 28.8 - 24.4 = 4.4$. For a policy $3 \in \hat{\boldsymbol{A}}_1(X^2)$, $\Delta_1^3 U_2(X^2) = U_2(X^2) - U_2(X^5) = 28.8 - 16.2 = 12.6$. Therefore, from $\min_{d \in \hat{\boldsymbol{A}}_1(X^2)\backslash 0} \Delta_1^d U_2(X^2) = 4.4 \leq r_1^3 = 10$, $d^* = 1$ is demonstrated. Accordingly, the maximum expected revenues and the optimal policies for all states in all time periods can be calculated.

**Table 3.1.** Arrival rate and fare for $p$ in $n$.

| | $\lambda_p^n$ | | | $r_p^n$ | | |
|---|---|---|---|---|---|---|
| $n\backslash p$ | 1 | 2 | 3 | 1 | 2 | 3 |
| 1 | 0.2 | 0.3 | 0.5 | 10 | 20 | 30 |
| 2 | 0.4 | 0.3 | 0.2 | 10 | 20 | 30 |
| 3 | 0.4 | 0.3 | 0.2 | 10 | 20 | 30 |
| 4 | 0.3 | 0.2 | 0.1 | 10 | 20 | 30 |

**Table 3.2.** The maximum expected revenues $U_n(X^k)$ for $n$ and $k$.

| $k\backslash n$ | 4 | 3 | 2 | 1 | 0 |
|---|---|---|---|---|---|
| 1 | 41.61 | 41.08 | 36.00 | 23.00 | 0.00 |
| 2 | - | 32.32 | 28.80 | 23.00 | 0.00 |
| 3 | - | 19.40 | 24.40 | 23.00 | 0.00 |
| 4 | - | 25.84 | 18.00 | 8.00 | 0.00 |
| 5 | - | 21.70 | 16.20 | 8.00 | 0.00 |
| 6 | - | - | 6.00 | 2.00 | 0.00 |
| 7 | - | 16.04 | 13.20 | 8.00 | 0.00 |
| 8 | - | - | 5.20 | 2.00 | 0.00 |
| 9 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |

**Table 3.3.** The optimal policies for $n$ and $X^k$.

| | $p=1$ | | | | $p=2$ | | | | $p=3$ | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $k\backslash n$ | 4 | 3 | 2 | 1 | 4 | 3 | 2 | 1 | 4 | 3 | 2 | 1 |
| 1 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 3 | 3 | 3 | 3 |
| 2 | - | 1 | 1 | 1 | - | 0 | 0 | 3 | - | 3 | 3 | 3 |
| 3 | - | 0 | 0 | 3 | - | 3 | 0 | 3 | - | 3 | 3 | 3 |
| 4 | - | 2 | 2 | 2 | - | 2 | 2 | 2 | - | 0 | 0 | 0 |
| 5 | - | 1 | 1 | 1 | - | 2 | 2 | 2 | - | 0 | 0 | 0 |
| 6 | - | - | 1 | 1 | - | - | 0 | 0 | - | - | 0 | 0 |
| 7 | - | 2 | 2 | 2 | - | 2 | 2 | 2 | - | 0 | 0 | 0 |
| 8 | - | - | 1 | 1 | - | - | 0 | 0 | - | - | 0 | 0 |
| 9 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

### 3.4.2 Counter-example

Note that if request $p$ is accepted, then all optimal policies in Table 3.3 take the smallest $d$. However, this feature is not held in the counter-example which is shown in this subsection. A data-set of this counter-example is $\overline{C} = 6, N = 3, \overline{P} = 3$ and $X_4 = \{(0, 1, 1, 0, 0, 0)\}$. Arrival rates and fares of this counter-example are shown in Table 3.4. Notations of elements of the state space are the same as the ones that was mentioned in the previous example.

Notice that the booking request $p = 2$ is accepted for all states in $n = 1, 2$ because the request $p = 2$ must arrive. Therefore, the maximum expected revenues can be easily calculated until $n = 2$. Since $\Delta_1^2 U_2(X^1) = U_2(X^1) - U_2(X^2) = 40 - 20 = 20$ and $\Delta_1^3 U_2(X^1) = U_2(X^1) - U_2(X^4) = 40 - 40 = 0$, the optimal policy for $p = 1$ and $X^1 \in X_3$ is $d^* = 3$. Thus, it shows non-monotonicity of $\Delta_p^d U_n(X)$ in $d (\neq 0)$. This counter-example indicates that Algorithm 3 is needed for calculating optimal policies and maximum expected revenues.

**Table 3.4.** Arrival rate and fare for $p$ in $n$.

| | $\lambda_p^n$ | | | $r_p^n$ | | |
|---|---|---|---|---|---|---|
| $n \backslash p$ | 1 | 2 | 3 | 1 | 2 | 3 |
| 1 | 0.0 | 1.0 | 0.0 | 10 | 20 | 30 |
| 2 | 0.0 | 1.0 | 0.0 | 10 | 20 | 30 |
| 3 | 0.4 | 0.3 | 0.2 | 10 | 20 | 30 |

## 3.5 Application of Single Line Seats Model

The single line seats model is applicable to a multi-line case if each line is not distinguished as shown in Figure 3.3. We can see this case as a single line seats model with any initial state. Thus,

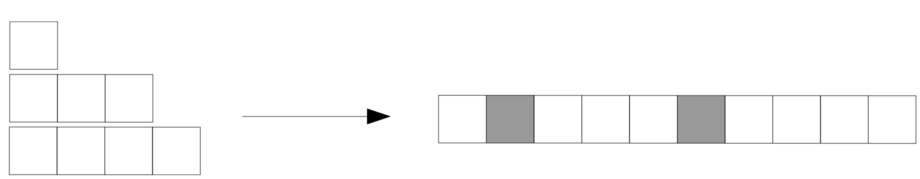

**Figure 3.3.** An example of modifying three lines to single line.

we can calculate optimal allocating position by using Algorithm 3. Further, if one fare class links to one seat, and arrivals of requests for each fare class are independent in different fare classes, then we can solve optimal policies for each fare class. Figure 3.4 shows an example where seats with multiple lines and multiple fare classes are modified to adopt the single line seats model.

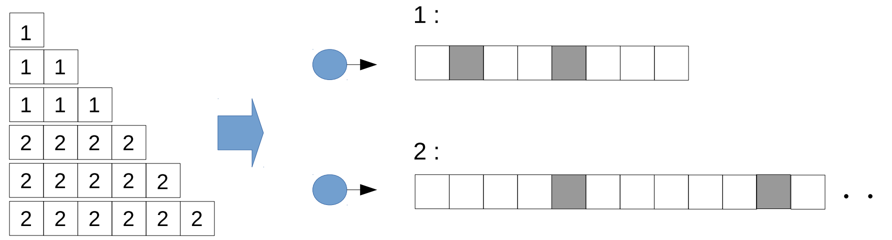


**Figure 3.4.** Modifying multiple line with two types fare classes.

## 3.6 Conclution

In this chapter, the single line seats model was introduced. Proposition 3.2 shows that we should allocate ends of vacant seats block to arriving request with any fare class and any size in order to obtain the maximum expected revenue. From the property, the relationship between the single line seats model and a basic dynamic model in [26] is revealed. Further, we indicate the conditions which make possible to apply the single line seats model for multiple lines with multiple fare classes.

From Remark 3.3, we can find that the single line seats is a extended one from basic dynamic models. However, customers cannot select their seating positions in the model suggested in this chapter. Nowadays, we can often select seating positions when we book seats because of development of online reservation system. In the next chapter, we mention optimal control for booking requests if arriving customers can choose their seating positions.

# List of Symbols for Chapter 4

$n$: remaining time to the terminal time

$N$: size of time period

$\lambda$: arrival rate

$T_0$: set of time periods $\{0, 1, \cdots, N\}$

$T_1$: set of time periods $\{1, \cdots, N\}$

$m$: the number of size of the longest segment

$M$: set of segments $\{1, \cdots, m\}$

$c$: initial state in time $N$; $c = (c_1, \cdots, c_m)^T$

$X$: state space

$(a, b) \in \Omega(x)$: possible choice set for $x \in X$ where $a$ and $b$ indicates a size of a segment and an index of the segment $a$, respectively

$S \subseteq \Omega(x)$: offer set

$\Omega$: all possible choice set; $\Omega = \left\{(a, b) | a \in M, b \in \mathbb{Z}_+, 0 < b \leq \frac{a+1}{2}\right\}$

$P_{(a,b)}(S)$: probability which is that an arriving customer decides to book the position $(a, b) \in S$ when offer set is $S$

$P_0(S)$: probability of no-purchase when offer set is $S$

$A_{(a,b)}$: changes of the numbers of segments when a customer chooses $(a, b) \in \Omega$; $A_{(a,b)} = \boldsymbol{e}_{b-1} + \boldsymbol{e}_{a-b} - \boldsymbol{a}$

$A$: $A = (A_{(a,b)})_{(a,b)\in\Omega}$

$U_n(x)$: maximum expected revenue which a facility obtains optimally operating from $n$ to 0, given the state $x \in X$ in $n \in T_0$

$\Delta_{(a,b)}U_n(x)$: opportunity cost of the seat $(a, b)$ at the state $x$ in time $n$

$R(S)$: expected revenue when offer set is $S$

$P(S)$: probability vector when offer set is $S$; $P(S) = (P_{(a,b)}(S))_{(a,b)\in\Omega}$

$Q(S)$: expected changing capacities when offer set is $S$; $Q(S) = AP(S)$ and $Q(S) = (Q_1(S), \cdots, Q_m(S))$

$U^{CDLP}$: maximum expected revenue obtained from CDLP

$\pi$: estimate of the marginal value of capacity on each segment; $\pi = (\pi_1, \cdots, \pi_m)^T$

$\sigma$: estimate of the marginal value of time

$\beta$: heuristic parameter in decomposition approximation method; $0 \le \beta \le 1$

$y$: binary vector which indicates whether positions are available or not

$v_{(a,b)}$: customer's preference of purchasing the position $(a, b) \in \Omega$

$v_0$: customer's preference of no-purchase

# Chapter 4

# Choice-based Seating Position Model

## 4.1 Introduction

Incorporating customer's purchasing behavior into dynamic models has been researched for decades. Talluri and van Ryzin [37] suggested a single-leg model with customer's behavior, and showed some properties for the model. In NRM, Gallego et al. [13], Zhang and Cooper [42], Liu and van Ryzin [27], Bront et al. [8] and etc. studied NRM model with customer's behavior to reveal structural properties and efficiently calculate approximate solutions. The NRM model with customer's behavior is called *choice-based NRM model*. Recently, Sierag et al. [33] suggested choice-based NRM model with overbooking and cancellation. Topics or models with customer's behavior are summarized in [32] as overview.

In this chapter, we show a dynamic model with seating position and customer's behavior by using the approaches of choice-based NRM. The dynamic model has multiple lines and multiple fares where each fare links to each seat.

## 4.2 Formulation for Choice-based Seating Position Model

Consider a facility with multiple seat lines. Assume that these lines do not distinguished. In addition, each fare class links to a seat, and arrivals of requests for each fare class are independent in different fare classes. From these assumptions, we can line seats up at each fare class and deal with the problem at each fare class. Hence, we consider the problem with only one fare class.

Booking horizon is sufficiently discretized into $n = 1, \cdots, N$. The times $n$ indicate remaining time to the terminal time $n = 0$. Assume that size of requests has one, and no more than one request arrives at $n$. Let $\lambda$ be rate of the requests which is dependent in times, and set $T_0 = \{0, \cdots, N\}$and $T_1 = \{1, \cdots, N\}$

States of the facility are shown by the numbers of adjacent vacant seats. We call this vacant seats a segment. If the segment have $n$ seats, we call the size of the segment $n$. The left side and right side of the segments are not distinguished. Let $m$ be the number of size of the longest segment. Set $M = \{1, \cdots, m\}$. Therefore, $c = (c_1, \cdots, c_m)^T$ stands for an initial state of the facility in time $N$ where $T$ means transpose, and $c_i$ indicates the initial number of segments of size $i$. State space is defined as

$$X = \left\{ (x_1, \cdots, x_m) | 0 \le x_i \le \sum_{k=1}^{m} c_k \lfloor \frac{k+1}{i+1} \rfloor, x_i \in \mathbb{Z}_+, i \in M \right\}.$$

where $\lfloor \frac{k+1}{i+1} \rfloor$ means how many segments of size $i$ is generated from a segment of size $k$. In addition, note that we do not use the assumption which is that one request arrives in time $n$ at most, to avoid complication.

In this model, if arriving customers book seats, then they select which segment and which position. Choices that a system of the facility can offer are defined as below.

$$\Omega(x) = \left\{ (a, b) | x_a > 0, b \in \mathbb{Z}_+, 0 < b \le \frac{a+1}{2}, a \in M \right\}, x \in X.$$

The empty set $\emptyset \subseteq \Omega(x), x \in X$ indicates not to offer seating positions for arriving requests, which is "deny" in dynamic models without customer's behavior. For each $x \in X$, $\Omega(x)$ is called the possible choice set. The system of the facility decides a subset of the possible choice set $S \subseteq \Omega(x), x \in X$ on a beginning time point of each time duration. The subset $S$ is called offer set. Arriving customers choose their booking positions from the offer sets which are decided by the system of the facility. The system aims to maximize expected revenue over booking horizon by deciding offer sets at each time and states.

Customers' behavior in this model is defined discretely, which can also be seen in [27, 37]. Arriving customers probabilistically choose their seating positions. Let $P_{(a,b)}(S)$ be a probability which is that an arriving customer decides to book the position $(a, b) \in S$ when $S \subseteq \Omega(x), x \in X$ is offered. Set $P_0(S) = 1 - \sum_{(a,b)\in S} P_{(a,b)}(S), S \subseteq \Omega(x), x \in X$, and $P_0(S)$ indicates a probability of no-purchase of customer's behavior when $S \subseteq \Omega(x), x \in X$ is offered.

Then, consider changes of the numbers of segments if an arriving customer chooses $(a,b) \in S \subseteq \Omega(x), x \in X$. When the customer chooses the segment with size $a$ and its index $b$, the change of state is $\boldsymbol{e}_{b-1} + \boldsymbol{e}_{a-b} - \boldsymbol{e}_a$, which means that the segment with size $a$ is split into a segments with size $b-1$ and $a-b$, where $\boldsymbol{e}_i$ indicates $i$-th unit vector for each $i \in M$, and $\boldsymbol{e}_0$ indicates zero vector. Set $A_{(a,b)} = \boldsymbol{e}_{b-1} + \boldsymbol{e}_{a-b} - \boldsymbol{e}_a$, and it is the changes of the numbers of segments when an arriving customer chooses $(a,b)$. In addition, define the matrix $A = (A_{(a,b)})_{(a,b)\in\Omega}$ where

$$\Omega = \left\{(a,b) | a \in M, b \in \mathbb{Z}_+, 0 < b \leq \frac{a+1}{2}\right\}.$$

Using dynamic programming, let $U_n(x)$ be the maximum expected revenue which the facility obtains optimally operating from $n$ to 0, given the state $x$ in time $n$.

$$\begin{aligned} U_n(x) = \max_{S \subseteq \Omega(x)} \Bigg\{ \lambda \sum_{(a,b)\in S} & P_{(a,b)}(S)(r + U_{n-1}(x + A_{(a,b)})) \\ & + (\lambda P_0(S) + (1-\lambda))U_{n-1}(X) \Bigg\}, n \in T_1, x \in X. \end{aligned} \tag{4.1}$$

Boundary conditions are $U_0(x) = 0, U_{N+1}(x) = 0, x \in X$, $U_n(0) = 0$ and $U_n(x) = 0, x \notin X, n \in T_1$. (4.1) is rewritten by $P_0(S) = 1 - \sum_{(a,b)\in S} P_{(a,b)}(S)$ as the following equations.

$$\begin{aligned} U_n(x) = \max_{S \subseteq \Omega(x)} \Bigg\{ \lambda \sum_{(a,b)\in S} P_{(a,b)}(S)(r - \Delta_{(a,b)} U_{n-1}(x)) \Bigg\} + U_{n-1}(x), \\ n \in T_1, x \in X \end{aligned} \tag{4.2}$$

where $\Delta_{(a,b)} U_n(x) = U_n(x) - U_n(x + A_{(a,b)})$ which means opportunity cost of a seat $(a,b)$ at the state $x$ in time $n$.

## 4.3 Approximation Model

In (4.2) using dynamic programming, it is obviously difficult to compute if $m$ and the numbers of initial segments enlarge by the curse of dimensionality mentioned in Section 2.1. Several approximation methods for this problem in NRM have been studied. Among them, some approximation methods have been suggested for the choice-based NRM models. Gallego et al. [13] suggested Choice-based Deterministic Linear Programming (CDLP) which utilizes DLP (Deterministic Linear Programming), which is the one of the approximation methods in NRM. Further, Liu and van Ryzin [27] suggested decomposition approximation method which is from dynamic programming decomposition method, which is a traditional approximation method in NRM. In this section, we apply CDLP and decomposition approximation method to the choice-based seating position model.

### 4.3.1 Choice-based Deterministic Linear Programming (CDLP)

Let the numbers of segments (means capacity) be continuous, and arrival rates be deterministic number, similar to Gallego et al. [13] and Liu and van Ryzin [27].

We set

$$R(S) = \sum_{(a,b)\in S} rP_{(a,b)}(S), S \subseteq \Omega(x), x \in X,$$

which means the expected revenue if offer set is $S$. Define $P(S) = (P_{(a,b)}(S))_{(a,b)\in\Omega}$. In addition, let $Q(S)$ be the expected changing capacities if offer set is $S$. Since the capacities is continuous,

$$Q(S) = AP(S)$$

where $Q(S) = (Q_1(S), \cdots, Q_m(S))$, and $Q_i(S)$ means the expected changing number of the segment with size $i$ if $S$ is offered.

Then, if any subsets are generated from $\Omega$, impossible combination of offer set would be created on actual transition from $c$. Therefore, we make the assumption for initial state $c$ as below.

**Assumption 4.1.** *On transition from a given initial state* $c$ *over time horizon, there exists states* $x^+ = (x_i)_{i\in M} \in X$*, such that* $x_i > 0, i \in M$

We do not consider not to move to the states $x^+$ by deficiency of time because we identify that $N$ is sufficiently large. We can see whether $c$ satisfies the assumption if we trace transitions of states. However, we can confirm whether $c$ satisfies a sufficient condition for the assumption from the simple fact which is that a segment $\beta$ is increased by one at most if a segment $\alpha(> \beta)$ is consumed. We propose an algorithm to confirm that. The order of the algorithm is $O(m^2)$. In

Algorithm 4, suppose that $m > 1$.

**Algorithm 4** An algorithm to confirm whether $c$ satisfies Assumption 4.1.

```
Input c
flag ← False
for i = 1 to m − 1 do
    if there exists j such that c_j ≠ 0, i < j and c_i = 0, then
        j* ← min{j}
        c_{j*} ← c_{j*} − 1
    end if
end for
if c_m ≠ 0 then
    flag ← True
end if
Output flag
```

Let $t(S)$ be the total number of time periods in which $S$ is offered. Then, we allow $t(S)$ to be continuous, which means that we can use the set $S$ for some fraction of a time period. Note that the sequence to offer $S$ is arbitrary.

Then, we obtain the maximum expected revenue $U^{CDLP}$ by CDLP.

$$
\begin{aligned}
U^{CDLP} = \max \sum_{S \subseteq \Omega} \lambda R(S) t(S) \\
\text{s.t. } \; 0 \leq c + \sum_{S \subseteq \Omega} \lambda Q(S) t(S) \\
\sum_{S \subseteq \Omega} t(S) \leq N \\
t(S) \geq 0, \forall S \subseteq \Omega.
\end{aligned} \tag{4.3}
$$

The dual problem of (4.3) is the following

$$
\begin{aligned}
\min \; & \pi^T c + T\sigma \\
\text{s.t. } \; & -\lambda \pi^T Q(S) + \sigma \geq \lambda R(S), \forall S \subseteq \Omega
\end{aligned} \tag{4.4}
$$

$$\pi \geq 0, \sigma \geq 0.$$

$\pi$ and $\sigma$ are dual variables corresponding to the first and second constraint equations in (4.3) where $\pi = (\pi_1, \cdots, \pi_m)^T$. From sensitivity analysis (referring [7, 17]) for (4.4), we can instinctively see that $\pi$ means an estimate of the marginal value of capacity on each segment, and $\sigma$ means an estimate of the marginal value of time.

Similar to Liu and van Ryzin [27], we mention the expected revenue which is obtained from initial state $c$ and time $N$. Define $\mu$ as control policy, which maps states to control actions (offer sets). $S_\mu(n|\mathcal{F}_n)$ is an action in time $n$ under the policy $\mu$ where $\mathcal{F}_n$ indicates the history of the system up to time $n$. To simplify notations, we omit $\mathcal{F}_n$ in the following sections. $N(S_\mu(n))$ denotes a $|\Omega|$-dimensional random vector which indicates the number of position purchased in time $n$ under the policy $\mu$. $N_{(a,b)}(S_\mu(n)) = 1, (a,b) \in \Omega$ means a sale of the position $(a,b)$ and $N_{(a,b)}(S_\mu(n)) = 0, (a,b) \in \Omega$ means no sale of the the position $(a,b)$. $\mathcal{M}$ denotes the class of all admissible policies. Noting that $\sum_{n=1}^{N} AN(S_\mu(n))$ is the changing quantities of the segments from $N$ to 1,

$$0 \leq c + \sum_{n=1}^{N} AN(S_\mu(n)) \quad \text{(a.s.)}$$

is satisfied.

From these notations, (4.2) is denoted as the general form;

$$\begin{aligned} U^* = \max_{\mu \in \mathcal{M}} E &\left[ r \sum_{n=1}^{N} e^T N(S_\mu(n)) \right] \\ \text{s.t. } 0 \leq c &+ \sum_{n=1}^{N} AN(S_\mu(n)) \quad \text{(a.s.)} \\ S_\mu(n) &\subseteq \Omega, n \in T_1, \end{aligned} \tag{4.5}$$

where $e = (1, \cdots, 1)^T$. We can obtain the following proposition which indicates the upper bound of (4.2) from the similar way in Liu and van Ryzin [27].

**Proposition 4.2.** $U^* \leq U^{CDLP}$.

***Proof:*** Let $S_{\mu^*}(n), n \in T_1$ be optimal controls. $0 \leq c + \sum_{n=1}^{N} AN(S_{\mu^*}(n))$ since $\mu^*$ is admissible policy. Then, set

$$t_{\mu^*}(S) = E\left[\sum_{n=1}^{N} 1_{S_{\mu^*}(n)}(S)\right]$$

and $t_{\mu^*}(S)$ is the expected total time in which $S$ has been offered under the policy $\mu^*$ where $1_{S_{\mu^*}(n)}(S), S \in \Omega$ is indicator function as

$$1_{S_{\mu^*}(n)}(S) = \begin{cases} 1 & S = S_{\mu^*}(n) \\ 0 & \text{otherwise.} \end{cases}$$

From Wald's equation (see p.521 in [12]),

$$\sum_{n=1}^{N} E\left[N_{(a,b)}(S_{\mu^*}(n))\right] = \sum_{S \subseteq \Omega} \lambda P_{(a,b)}(S) t_{\mu^*}(S)$$

and $0 \leq c + \sum_{S \subseteq \Omega} \lambda AP(S) t_{\mu^*}(S)$ are obtained. In addition, $U^* = \sum_{n=1}^{N} re^T E\left[N(S_{\mu^*}(t))\right] = \sum_{S \subseteq \Omega} \lambda re^T P(S) t_{\mu^*}(S)$. From definitions of $Q(S)$ and $R(S)$, we can find that $t_{\mu^*}(S)$ is a feasible solution for the problem (4.3). Hence $U^* \leq U^{CDLP}$ is shown. ■

A definition of efficient set in choice-based NRM model which is shown in [27] is ineffective in this model. The definition of efficient set is shown as a reference.

**Definition 4.3.** For an offer set $K$, if there exists a set of convex weights $\alpha(S), S \subseteq \Omega$ satisfying $\sum_S \alpha(S) = 1$ and $\alpha(S) \geq 0, \forall S \subseteq \Omega$ such that

$$R(K) < \sum_S \alpha(S) R(S)$$
$$Q(K) \geq \sum_S \alpha(S) Q(S),$$

then the set $K$ is said to be inefficient. If there does not exit the weight, $K$ is said to be efficient.

Optimality of the efficient set cannot be applied for choice-based seating position model. The optimality are shown as several forms in [27, 37, 42]. As a reference, we indicate the proposition shown by [27].

**Proposition 4.4** (Liu and Ryzin [27]). *A set $K$ is efficient if and only if, for some $\xi = (\xi_1, \cdots, \xi_m)^T \geq 0$, $K$ is the optimal solution to*

$$\max_S \{R(S) - \xi^T Q(S)\}.$$

Noting $\xi \leq 0$ in choice-based seating position model, we can use this proposition for the only case $\xi = 0$. Therefore, the definition of efficient set is hardly used to this model.

We apply column generation (specifically cutting plane method [7]) to (4.3), referring to [13,27]. Let $\hat{S} \subseteq \Omega$. The reduced CDLP for the limited subset $\hat{S}$ is

$$\begin{aligned} U^{CDLP}(\hat{S}) = \max & \sum_{S \in \hat{S}} \lambda R(S) t(S) \\ \text{s.t.} \quad & -\sum_{S \in \hat{S}} \lambda Q(S) t(S) \leq c \\ & \sum_{S \in \hat{S}} t(S) \leq N \\ & t(S) \geq 0, \forall S \in \hat{S}, \end{aligned} \tag{4.6}$$

where $\pi$ and $\sigma$ are the optimal solution of the dual problem for (4.6). These dual variables $\pi$ and $\sigma$ are corresponding to the first and second constraint equations of (4.6), respectively. We solve the following sub-problem to produce whether the dual solution is feasible for (4.3).

$$\max_S \lambda(R(S) + \pi^T Q(S)) - \sigma. \tag{4.7}$$

If the optimal value of (4.7) is non-positive, then the dual solution is feasible and optimal solution for (4.3). If the optimal value of (4.7) is positive, then we include the solution to $\hat{S}$ and recalculate (4.6).

### 4.3.2 Decomposition Approximation Method

The solutions given by CDLP are times to allocate to each offer set. That has problem which is that a sequence of offering the sets through booking time is arbitrary. This means that we cannot identify what states and times to apply the offer sets, even though we can see how much time to allocate to each offer set. To resolve this problem of CDLP, Liu and van Ryzin [27] suggested decomposition approximation method for choice-based NRM models. In the rest of this chapter, the decomposition approximation method improved by Bront et al. [8] is applied to the choice-based seating position model. In the decomposition approximation method, we decompose (4.2) in each segment by using its marginal value which is given as the dual solution $\pi = (\pi_1, \cdots, \pi_m)^T$, that is,

$$U_n(x) \approx \hat{U}_n^i(x_i) + \sum_{l \neq i} \pi_l x_l, n \in T_1, i \in M, x = (x_1, \cdots, x_m) \in X. \tag{4.8}$$

**Phase 1: calculate one-dimensional dynamic programming**

We calculate the one-dimensional dynamic programming $\hat{U}_n^i(x_i), n \in T_1, i \in M, x = (x_1, \cdots, x_m) \in X, (a, b) \in S \subseteq \Omega(x)$.

From (4.8),

$$\begin{aligned}\Delta_{(a,b)} U_n(x) &= U_n(x) - U_n(x + A_{(a,b)}) \\ &\approx \hat{U}_n^i(x_i) - \hat{U}_n^i(x_i + e_i^T A_{(a,b)}) - (\pi^T - \pi_i e_i^T) A_{(a,b)}, n \in T_1, i \in M, \\ &\quad x = (x_1, \cdots, x_m) \in X, (a, b) \in S \subseteq \Omega(x). \end{aligned} \tag{4.9}$$

is obtained. Using (4.8) and (4.9),

$$\hat{U}_n^i(x_i) = \max_{S \in \overline{S}_i(x_i)} \left\{ \lambda \sum_{(a,b) \in S} P_{(a,b)}(S)(r + (\pi^T - \pi_i e_i^T) A_{(a,b)} - \Delta_{(a,b)} \hat{U}_{n-1}^i(x_i)) \right\} + \hat{U}_{n-1}^i(x_i) \tag{4.10}$$

is produced from (4.2) where $\overline{S}_j(x_j) = \{S | S \subseteq \Omega(x), x_i = x_j, x \in X\}$. $\overline{S}_j(x_j)$ indicates the action space if the value of the $i$-th element of the state is $x_j$.

**Phase 2: find offer sets**

Finally, using $\hat{U}_n^i(x_i)$ calculated in Phase 1, we approximately calculate the deflection vector $\Delta U_n^i(x) = U_n(x) - U_n(x - \boldsymbol{e}_i)$ for each $i \in M$, $n \in T_1$, and $x \in X$, and find offer sets for each time $n \in T_1$ and state $x \in X$ by using the deflection vectors.

Consider a heuristic parameter $0 \leq \beta \leq 1$ and calculate

$$\Delta U_n^i(x) \approx \Delta \overline{U}_n^i(x) = \beta \Delta \hat{U}_n^i(x_i) + (1-\beta)\pi_i, i \in M, n \in T_1, x = (x_1, \cdots, x_m) \in X, \quad (4.11)$$

where $\Delta \hat{U}_n^i(x_i) = \hat{U}_n^i(x_i) - \hat{U}_n^i(x_i - 1)$ and $\Delta \hat{U}_n^i(0) = 0$. Using $\Delta \hat{U}_n^i(x)$, for each time $n \in T_1$ and state $x \in X$, solve

$$\max_{S \subseteq \Omega(x)} \left\{ \lambda \sum_{(a,b) \in S} P_{(a,b)}(S)(r - \Delta \overline{U}_{n-1}^T(x) A_{(a,b)}) \right\} \quad (4.12)$$

where $\Delta \overline{U}_n^T(x) = (\Delta \overline{U}_n^1(x), \cdots, \Delta \overline{U}_n^m(x))$. From this way, we can obtain offer sets for each time and state by decomposition approximation method.

## 4.4 Applying MNL choice model to customer's behavior

Remark that the approximation methods are effective ones for the curse of dimensionality. Specifically, the approach of CDLP is to resolve it by sacrificing sequences of offer sets, and the approach of decomposition approximation method is to reduce it by decomposing states. However, these methods do not resolve enlarging size of possible offer set. The numbers of seats of a fare class in theater, stadium, opera, Kabuki and etc. are actually very large. The dimension $m$ is derived from the number of seats at the longest line. Since the number of possible offer set is considerably effected by the dimension, it is obvious that the number of all subsets of $\Omega$ is exponentially increased. In the approximation methods, this problem appears in calculating (4.7), (4.10) and (4.12).

We show to able to resolve this problem under an assumption which is that customers' behavior depends on Multinomial Logit (MNL) choice model, which can be also seen in [13, 27]. What customers' behavior depends on MNL choice model means that coustomers have a preference (weight) for each position, and select a position depending by their preferences. In the following chapter, we consider that all customers treat all possible positions as their selections. Then, let

$y = (y_{(a,b)})_{(a,b)\in\Omega}$ be a binary vector where

$$y_{(a,b)} = \begin{cases} 1 & (a,b) \text{ is available,} \\ 0 & (a,b) \text{ is non-available,} \end{cases}$$

which indicates whether the position $(a,b)$ is available or not. Using the $y$, we define

$$P_{(a,b)}(y) = \frac{v_{(a,b)} y_{(a,b)}}{\sum_{(\alpha,\beta)\in\Omega} v_{(\alpha,\beta)} y_{(\alpha,\beta)} + v_0} \tag{4.13}$$

as a probability that a customer chooses the position $(a,b) \in \Omega$, where $v_{(a,b)} \geq 0$ and $v_0 > 0$ are preferences of purchasing the position $(a,b)$ and no-purchase, respectively. Applying the MNL choice model to (4.2), (4.7) and (4.12), we obtain

$$U_n(x) = \max_{y\in Y(x)} \frac{\lambda \sum_{(a,b)\in\Omega} (r - \Delta_{(a,b)} U_{n-1}(x)) v_{(a,b)} y_{(a,b)}}{\sum_{(a,b)\in\Omega} v_{(a,b)} y_{(a,b)} + v_0} + U_{n-1}(x),$$
$$n \in T_1, x \in X, \tag{4.14}$$

$$\lambda \max_{y\in\{0,1\}^{|\Omega|}} \frac{\sum_{(a,b)\in\Omega} (r + \pi^T A_{(a,b)}) v_{(a,b)} y_{(a,b)}}{\sum_{(a,b)\in\Omega} v_{(a,b)} y_{(a,b)} + v_0} - \sigma \tag{4.15}$$

and

$$\max_{y\in Y(x)} \left\{ \frac{\lambda \sum_{(a,b)\in\Omega} (r - \Delta_{(a,b)} \hat{U}_{n-1}(x)) v_{(a,b)} y_{(a,b)}}{\sum_{(a,b)\in\Omega} v_{(a,b)} y_{(a,b)} + v_0} \right\}, \tag{4.16}$$

respectively, where

$$Y(x) = \left\{ (y_{(a,b)})_{(a,b)\in\Omega} | y_{(a,b)} \in \{0, 1_{\Omega(x)}(a,b)\}, (a,b) \in \Omega \right\}, x \in X \tag{4.17}$$

and

$$1_{\Omega(x)}(a,b) = \begin{cases} 1 & (a,b) \in \Omega(x) \\ 0 & \text{otherwise.} \end{cases}$$

The following Proposition 4.5 is Proposition 6 in [27].

**Proposition 4.5** (Liu and van Ryzin [27]). *Consider a problem*

$$\max_{y \in \{0,1\}^{|\Omega|}} \frac{\sum_{(a,b)\in\Omega} \xi_{(a,b)} v_{(a,b)} y_{(a,b)}}{\sum_{(a,b)\in\Omega} v_{(a,b)} y_{(a,b)} + v_0}. \tag{4.18}$$

*When $\xi_{(a,b)}, (a,b) \in \Omega$ are ranked in a decreasing order, let $\xi_{[i]}$ be the $i$-th value, that is,*

$$\xi_{[1]} \geq \cdots \geq \xi_{[i]} \geq \cdots \geq \xi_{[|\Omega|]}.$$

*Then, there is a critical value $k^*, 1 \leq k^* \leq \Omega$ such that*

$$y^*_{(a,b)} = \begin{cases} 1 & \xi_{(a,b)} \geq \xi_{[k^*]}, \\ 0 & \xi_{(a,b)} < \xi_{[k^*]} \end{cases}$$

*where $y^* = (y^*_{(a,b)})$ is an optimal solution for the above problem (4.18).*

Furthermore, we show the proposition which is easily obtained from Proposition 4.5.

**Proposition 4.6.** *Given a state $x \in X$, consider the problem*

$$\max_{y \in Y(x)} \frac{\sum_{(a,b)\in\Omega} \xi_{(a,b)} v_{(a,b)} y_{(a,b)}}{\sum_{(a,b)\in\Omega} v_{(a,b)} y_{(a,b)} + v_0}. \tag{4.19}$$

*When $\xi_{(a,b)}, (a,b) \in \Omega$ are ranked in a decreasing order, let $\xi_{[i]}$ be the $i$-th value, that is,*

$$\xi_{[1]} \geq \cdots \geq \xi_{[i]} \geq \cdots \geq \xi_{[|\Omega(x)|]}.$$

*Then, there is a critical value $k^*, 1 \leq k^* \leq \Omega$ such that*

$$y^*_{(a,b)} = \begin{cases} 1 & \xi_{(a,b)} \geq \xi_{[k^*]}, 1_{\Omega(x)}(a,b) = 1 \\ 0 & \textit{otherwise} \end{cases}$$

*where $y^* = (y^*_{(a,b)})$ is an optimal solution for the above problem (4.19).*

***Proof:*** Consider the problem

$$\max_{y \in Y'(x)} \frac{\sum_{(a,b)\in\Omega(x)} \xi_{(a,b)} v_{(a,b)} y_{(a,b)}}{\sum_{(a,b)\in\Omega(x)} v_{(a,b)} y_{(a,b)} + v_0} \tag{4.20}$$

where

$$Y'(x) = \left\{ (y_{(a,b)})_{(a,b)\in\Omega(x)} | y_{(a,b)} \in \{0,1\}, (a,b) \in \Omega(x) \right\}, x \in X. \tag{4.21}$$

It is obvious that the problem (4.19) is the same as the problem (4.20) because terms of $(a, b)$ such that $1_{\Omega(x)}(a, b) = 0$ is zero in (4.19). Hence, Proposition 4.6 is obtained by applying Proposition 4.5 to the problem (4.20). ■

Proposition 4.5 is helpful to search optimal solutions of this model. From the proposition, we can solve (4.15) by calculating $|\Omega|$ patterns at most. However, we need to calculate $2^{|\Omega|}$ patterns without Proposition 4.6 in the worst case. Similarly, we can use Proposition 4.6 for effectively solving (4.14) and (4.16).

## 4.5 Numerical Example

In this section, we estimate the each approximation method by Monte Carlo simulation. Set $N = 100, \lambda = 0.3, r = 10$ and $c = (0, 0, 4, 4)$. For preferences, let $v_{(1,1)} = 0.5, v_{(2,1)} = 1.5, v_{(3,1)} = 2.0, v_{(3,2)} = 3.0, v_{(4,1)} = 2.5, v_{(4,2)} = 3.5$ and $v_0 = 2.0$. Configurations of the approximation methods are suggested as below.

DP: DP means that using offer sets which are obtained by (4.14).

CDLP-LX: Using solutions obtained by CDLP, we allocate offer sets with positive time to each occurred state in backward lexicographical order. The backward lexicographical order means that we allocate offer sets to each occurred state in backward order when we regard the offer sets as character strings and sort the strings in lexicographical order. For examples, when there are offer sets $\{(1, 1), (3, 1)\}$ and $\{(2, 1), (3, 1), (3, 2)\}$ with positive allocating time, we firstly select $\{(2, 1), (3, 1), (3, 2)\}$. In this way, we set that positions in each offer set have been sorted in lexicographical order.

CDLP-RND: We randomly choose an offer set from offer sets with positive allocating time.

DCOMP-0, DCOMP-0.5, DCOMP-1: DCOMP-0, DCOMP-0.5 and DCOMP-1 indicate to use offer sets which are computed by the decomposition approximation method with $\beta = 0$, $\beta = 0.5$ and $\beta = 1$, respectively.

FULL-OPEN: This indicates that we always open unbooked seats.

We trace paths of state from $n = N$ to $n = 0$ by 100 times, and we calculate averages of total obtained revenue for each history. The result is shown in Figure 4.1 where the vertical axis indicates the average revenue and the horizontal axis indicates the number of times of the paths.

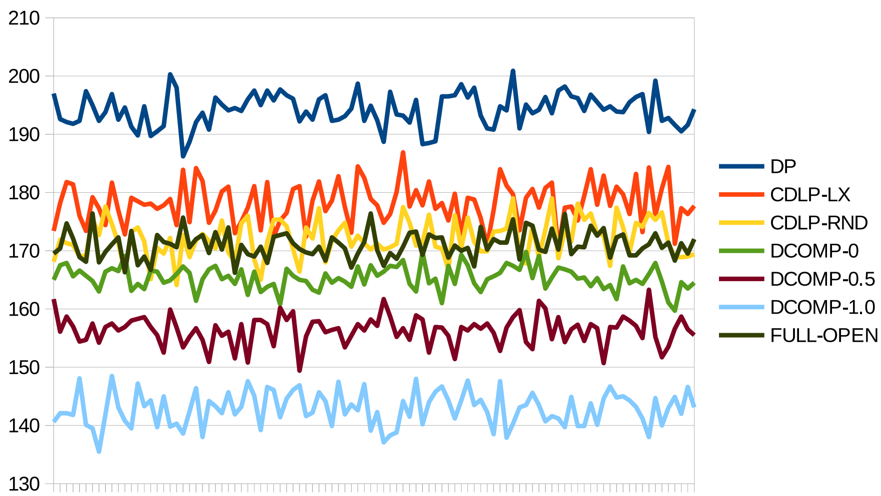


**Figure 4.1.** Average revenue for each approximation method.

In this result, we can see that CDLP-LX and DP make higher revenue than FULL-OPEN which is actually no-control. All decomposition approximation methods with $\beta = 0, 0.5$ and $1$ make lower revenue than FULL-OPEN. Note that differences among the methods are not generated if a customer's preference of no-purchase is zero. The reason is that it is optimal to accept all customers' requests if all arriving customers always book positions, since revenue obtained from each customer is the same. Therefore, the differences among the methods are derived from customers' behavior, that is, we can obtain higher revenue by controlling choices of seating position for customers even though fares for all seats are the same.

## 4.6 Conclusion

In this chapter, we proposed choice-based seating position model which deals with customers who choose their booking position or no-purchase on their preferences in a facility with multiple lines under some assumptions. In addition, we showed upper bound of the expected revenue and that we can apply techniques of choice-based NRM to the dynamic model with seating position. In the numerical example, it is indicated that we can obtain higher revenue by CDLP than FULL-OPEN.

If customers' behavior depends on MNL choice model, we can efficiently calculate optimal offer sets by Propositions 4.5 and 4.6. For the model by dynamic programming, we can reduce searching in action space and may prospect to obtain the highest revenue although the curse of dimensionality is not resolved. Using CDLP method, we can use linear programming and can efficiently solve the sub-problem by Proposition 4.5, that is, we can deal with large $m$, $c$ and $N$. Further, it is suggested that CDLP may be effective approximation methods for choice-based seating position model since there is a case in which offer sets obtained by CDLP can generate higher revenue than FULL-OPEN. On the other hand, in the decomposition approximation method, a part of procedures for finding optimal offer sets cannot be effectively calculated. Also, it is suggested that the method may be not effective in the numerical example. This result is different from ones in [27] and [8] which treat a basic choice-based NRM model. The reason may be that the elements of states in choice-based seating position model, means segment, are strongly related among segments with different size, which is different from normal RM models.

For the choice-based seating position model, there is the effective way to calculate offer sets if all customers depend on the same preference. Bront et al. [8] have presented that the sub-problem in the basic choice-based NRM model with multiple customer's segment is NP-hard if the choices of the customer's segments overlap with each other. They also suggested that the problem can be solved in practical time since the problem is conducted by a mixed integer programming. In this chapter, we do not concern multiple customer's segments with different preferences (for example, membership or non-membership who have different enthusiasm for services provided by facilities, which is opera or Kabuki). This issue which leads to include actual conditions to the choice-based seating position model may join to their works which consider the basic choice-based NRM model.

# Chapter 5

# Summary and Future issues

In the party mix problem, we provided some assumptions to simplify existing models, and showed the upgrade structure and the range of generating variations of optimal policies. However, it is mysterious that how the assumption of departure of customers depending on exponential distribution can apply to actual facilities. Distribution of the staying time, which is called meal duration, has not been focused, although fundamental statistics for that have been studied. If there are services where departures of customers can be approximated by exponential distribution, then we can achieve the maximum expected revenue in the services by deciding whether each arriving party is accepted to allocate the smallest table or not.

However, when there are customers hoping to sit in large tables (for example, they have many baggages), is it the best way to always allocate available smallest tables to the parties? Considering with the work in [15] which shows that customer satisfaction leads to increase the customer's loyalty for a facility, the relationship between the number of seats that a party has and the facility's expected revenue are regarded as the result under the condition in which customers make much account of comfort given by sitting in large table. Actually, the customers may be a part of people but not all, that is, there are many kinds of customers. As an issue relating to this problem, Wang et al. [41] reported that integration between RM and Customer Relationship Management (CRM) is a challenging for RM theory. Recently, Klein and Kolb [24] suggested a model with considering customers who change their behavior by a facility's decision-making. However, it is difficult to apply their approach to a dynamic model from a viewpoint of computational cost. Thus, the integration is still a future issue.

In the chapter 3, we proposed a dynamic model with seating positions, using an approach of formulation of pyramid game by Fujita [18]. In the model, we showed that allocating edges of a vacant seat block to an arriving party is optimal policy, in addition, the model is extended from a Lee and Hersh's model which is one of traditional dynamic models. From the result, we produced the algorithm which is used to calculate optimal policies and the maximum expected revenue. The model is simple. However, it stands for complex transitions of states which are derived from the condition that is a group is seated together, and can extensively apply to the case of multiple lines if the each line is not distinguished.

In the chapter 4, we consider to include customers' behavior for seating positions to the dynamic model, which can be seen in online reservation system. We assumed that each seat line is not distinguished, and one seat links to one fare class. Arriving customers decide whether to book which seating position or not by their preference for each seating position. In this model, action space is exponentially increased since the action of a system is a subset of possible seating positions. However, if the customers' behavior for choosing their seating position depends on MNL model, then we showed that the optimal offer set can be solved, effectively. This means that insights and results which is obtained by studies of choice-based NRM model, which have been studied for about the last decade, can be used to the choice-based seating position model. In addition, we can consider more extension for the model, for example, to consider arriving parties, customers with various behavior without MNL model, and etc. Unless the seating position is focused, this direction is not produced and do not lead to allow the industries, for example, opera, theater, sport stadium, Kabuki, and etc. to increase their revenue by controlling only offer set of seating position on their existing systems.

In this thesis, we focused the spatial conditions of seats on dynamic models, and new models and their properties have been shown. These results have theory of RM include new approach which is to consider congestion level and seating position. We can perceive new direction and insight in RM from these results.

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